251 Abstract Algebra - Midterm 1 - Solutions

Question 1

Let $\sigma \in S_8$ be the following permutation:

$$\begin{array}{ll} 1 \mapsto 3 & 5 \mapsto 2 \\ 2 \mapsto 4 & 6 \mapsto 6 \\ 3 \mapsto 8 & 7 \mapsto 7 \\ 4 \mapsto 5 & 8 \mapsto 1. \end{array}$$

(a) Find the cycle decomposition of σ and σ^{-1}.

(b) Find $|\sigma|$.

(c) Write σ as a product of (not necessarily disjoint) cycles of length 2.

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Solution.

(a) From the definition of σ, we see that

$$\sigma = (1\ 3\ 8)(2\ 4\ 5)$$

and

$$\sigma^{-1} = (1\ 8\ 3)(2\ 5\ 4).$$

(b) We know that $|\sigma|$ is the LCM of its cycle lengths in the cycle decomposition. In this case, those are 3, 3, 1, 1, and therefore $|\sigma| = 3$.

(c) We can work on the two disjoint cycles of length 3 separately, and see that

$$\sigma = (1\ 8)(1\ 3)(2\ 5)(2\ 4).$$

Question 2

Let H be a nonempty subset of a finite group G, and suppose that for all $x, y \in H$, we have $xy \in H$. Show that for all $x \in H$, we have $x^{-1} \in H$. (This is part of the proof of the Subgroup Criterion.)

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Solution. Since H is nonempty, we let $x \in H$. Since H is closed under multiplication, we have that $x, x^2, x^3, \ldots$ are also in H. Now H is a subset of a finite group G, and therefore H must have finitely many elements, and therefore there is repetition in the list $x, x^2, x^3, \ldots$. Suppose that $x^a = x^b$ with $a < b$, then rearranging gives $x^{b-a} = 1$. Since the order of x is defined as the smallest natural number n such that $x^n = 1$, we know that $n \leq b - a$ and therefore finite. Let $|x| = n$. Then we have $xx^{n-1} = 1$ and $x^{-1} = x^{n-1}$, which we have shown to be in H.

Question 3

For a group G and subset $A \subseteq G$, let $N_G(A)$ be the normalizer of A in G. Show that $N_G(A) \leq G$.

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Solution. We will show this via the Subgroup Criterion. Since 1 commutes with all elements of G, it commutes with all elements of A. Then $1a1^{-1} = a$ for all $a \in A$, and therefore $1A1^{-1} = A$. So we have that $1 \in N_G(A)$ and $N_G(A)$ is nonempty. Suppose that $x, y \in N_G(A)$. We have that $yAy^{-1} = A$. Rearranging gives $A = y^{-1}Ay$. Now, we see that

$$\begin{aligned}(xy^{-1})A(xy^{-1})^{-1} &= xy^{-1}Ayx^{-1} \\ &= x(y^{-1}Ay)x^{-1} \\ &= xAx^{-1} \\ &= A,\end{aligned}$$

as needed. Therefore, $xy^{-1} \in N_G(A)$ and we are done.

Question 4

Consider the dihedral group D_{2n} where $n = 2k$ is an even number:

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle.$$

(a) Show that the element $z = r^k$ commutes with all elements of D_{2n}.

(b) Show that z is the only non-identity element that commutes with all elements of D_{2n}.

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Solution.

(a) By the relations on D_{2n}, we have seen that any element of D_{2n} can be written in the form $s^i r^j$ with $0 \leq i \leq 1$ and $0 \leq j \leq n-1$. Therefore, if we can show that z commutes with both s^i and r^j we are done. If $i = 0$, then $s^i = 1$ and we are done. Otherwise

$$zs = r^k s = sr^{-k} = sr^{-k}1 = sr^{-k}r^n = sr^{-k}r^{2k} = sr^{-k} = sz.$$

Furthermore, we have

$$zr^j = r^k r^j = r^{k+j} = r^{j+k} = r^j r^k = r^j z,$$

as needed.

(b) Let $s^i r^j$ as before be an element of D_{2n} that is not the identity and not z. Suppose that $i = 0$. Then our element is r^j with $j \notin \{0, k\}$. Then

$$sr^j = r^{-j}s.$$

However, since $j \neq k$, we have that $r^{-j} = r^{n-j}$ with $1 \leq n-j \leq n-1$ and $n-j \neq j$. Since we know that the elements $1, r, r^2, \ldots, r^{n-1}$ are distinct we conclude that $r^j \neq r^{-j}$. Now suppose that $i = 1$ and $0 \leq j \leq n-1$. Then

$$rsr^j = sr^{-1}r^j = sr^{j-1}.$$

Since $r \neq 1$, we see that $r^{j-1} \neq r^j$. Therefore, our element $s^i r^j$ does not commute with all elements of D_{2n}.

Question 5

(a) For a group G acting on a set S. Let G_s be the stabilizer of $s \in S$ of the action. Show that $g \in G_s$ implies that $g^{-1} \in G_s$. (This is part of the proof of showing that the stabilizer is a subgroup of G.)

(b) Let a group G act on itself by conjugation: let $g \cdot h = ghg^{-1}$ for all $g, h \in G$. For a given element $a \in G$, describe the stabilizer G_a in terms of normalizers/centralizers/center.

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Solution.

(a) Suppose that $g \in G_s$. Then

$$s = 1 \cdot s = (g^{-1}g) \cdot s = g^{-1} \cdot (g \cdot s) = g^{-1} \cdot s,$$

by the definition of a group action and the fact that $g \in G_s$. Therefore, $g^{-1} \in G_s$.

(b) We have

$$G_a = \{g \in G \mid g \cdot a = a\} = \{g \in G \mid gag^{-1} = a\} = C_G(a),$$

by definition of the centralizer of a subset.