251 Abstract Algebra - Midterm 1 - Practice - Solutions

Question 1

Let $\sigma \in S_6$ be the following permutation:

$$\begin{array}{ll} 1 \mapsto 3 & 4 \mapsto 1 \\ 2 \mapsto 4 & 5 \mapsto 6 \\ 3 \mapsto 2 & 6 \mapsto 5. \end{array}$$

(a) Find the cycle decomposition of σ and σ^{-1}.

(b) Find $|\sigma|$.

(c) Consider the element $\tau = (1\ 2\ 3)$. Find two elements $\tau_1, \tau_2 \in S_6$ such that $|\tau_1| = |\tau_2| = 2$ and $\tau = \tau_1\tau_2$.

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Solution.

(a) From reading the mapping (backwards and forwards), we see that

$$\sigma = (1\ 3\ 2\ 4)(5\ 6)$$
$$\sigma^{-1} = (1\ 4\ 3\ 2)(5\ 6).$$

(b) We can do this by writing out $\sigma, \sigma^2, \ldots$, but we have also learned that the order of an element in S_n is the LCM of the lengths of the cycles in its cycle decomposition. Therefore, we see that $|\sigma| = lcm(4, 2) = 4$.

(c) There are multiple solutions to this, which we can find by trial and error:

$$\tau = (1\ 2\ 3) = (1\ 3)(1\ 2) = (2\ 3)(1\ 3) = (1\ 2)(2\ 3).$$

As a general question: try to write a single cycle $(1\ \ldots\ k)$ as a product of transpositions.

Question 2

Let H be a nonempty subset of a group G, and suppose that for all $x, y \in H$, we have $xy^{-1} \in H$. Show that for all $x \in H$, we have $x^{-1} \in H$. (This is part of the proof of the Subgroup Criterion.)

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Solution. First, we know that H is nonempty, so we can let x be an element of H (which we know exists). If we let $x = y$, we obtain that $xx^{-1} = 1 \in H$. So, H contains the identity. This means that for any $x \in H$, we have $1, x \in H$ and therefore $1x^{-1} = x^{-1} \in H$.

Question 3

For a group G and subset $A \subseteq G$, let $C_G(A)$ be the centralizer of A in G, and let $Z(G)$ be the center. Show that $C_G(Z(G)) = G$.

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Solution. The centralizer is defined as

$$C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\},$$

and the center as

$$Z(G) = \{h \in G \mid hbh^{-1} = b \text{ for all } b \in G\}.$$

If $a \in Z(G)$ and any $g \in G$, then we know that $aga^{-1} = g$ by the definition of $Z(G)$. However we can rewrite this as $ag = ga$ which implies $a = gag^{-1}$. Since this holds for every $g \in G$, we have

$$C_G(Z(G)) = G.$$

Question 4

Find an injective homomorphism $\phi : C_3 \to S_4$, by giving an explicit injective map and showing that it is indeed a homomorphism.

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Solution. The group $C_3 = \{1, r, r^2\}$ consists of one element of order 3 and its powers. So, a good place to start is an element in S_4 of order 3, such as (1 2 3). We let $\phi : C_3 \to S_4$ be defined by

$$\begin{aligned} 1 &\mapsto 1 \\ r &\mapsto (1\ 2\ 3) \\ r^2 &\mapsto (1\ 2\ 3)^2 = (1\ 3\ 2). \end{aligned}$$

To show that this is a homomorphism, we need to show that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in C^3$.

$$\begin{aligned} \phi(1r) &= 1(1\ 2\ 3) = (1\ 2\ 3) = \phi(r) \\ \phi(r1) &= (1\ 2\ 3)1 = (1\ 2\ 3) = \phi(r) \\ \phi(1r^2) &= 1(1\ 3\ 2) = (1\ 3\ 2) = \phi(r^2) \\ \phi(r^2 1) &= (1\ 3\ 2)1 = (1\ 3\ 2) = \phi(r^2) \\ \phi(rr^2) &= (1\ 2\ 3)(1\ 3\ 2) = 1 = \phi(1) \\ \phi(r^2 r) &= (1\ 3\ 2)(1\ 2\ 3) = 1 = \phi(1), \end{aligned}$$

as needed.

Question 5

(a) For a group G acting on a set S. Let G_s be the stabilizer of $s \in S$ of the action. Show that G_s is closed under multiplication. (This is part of the proof of showing that the stabilizer is a subgroup of G.)

(b) Let D_8 act on the corners of a square in the usual way. Number the corners in clockwise order as $\{1, 2, 3, 4\}$. Then $\sigma_r = (1\ 2\ 3\ 4)$ and $\sigma_s = (1\ 2)(3\ 4)$. Find $(D_8)_1$, i.e. the stabilizer of 1 in D_8.

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Solution.

(a) We know (by the definition of a group action) that the identity of G is always in the stabilizer of s, so G_s is nonempty. Suppose that $g, h \in G_s$. Then

$$(gh) \cdot s = g \cdot (h \cdot s) = g \cdot s = s,$$

by the definition of the group action and the fact that $g, h \in G_s$. Therefore, $gh \in G_s$.

(b) The only rotation that leaves any corner in place is the identity. Of the reflections, the only one that leaves a particular corner in place is one whose axis passes through that corner. In this case, we can write out the permutations σ_g for all $g \in D_8$ explicitly:

$$\begin{aligned}
&\sigma_1 = 1 && \sigma_s = (1\ 2)(3\ 4)\\
&\sigma_r = (1\ 2\ 3\ 4) && \sigma_{sr} = (1\ 2)(3\ 4)(1\ 2\ 3\ 4) = (2\ 4)\\
&\sigma_{r^2} = (1\ 2\ 3\ 4)^2 = (1\ 3)(2\ 4) && \sigma_{sr^2} = (1\ 2)(3\ 4)(1\ 3)(2\ 4) = (1\ 4)(2\ 3)\\
&\sigma_{r^3} = (1\ 2\ 3\ 4)^3 = (1\ 4\ 3\ 2) && \sigma_{sr^3} = (1\ 2)(3\ 4)(1\ 4\ 3\ 2) = (1\ 3).
\end{aligned}$$

From this, it is even easier to see that the set of permutations that fix 1 is $\{1, \sigma_s r\}$ and therefore $G_s = \{1, sr\}$.