251 Abstract Algebra - Final - Practice - Solutions

Question 1

Prove that σ^2 is an even permutation for every $\sigma \in S_n$.

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Solution. By Proposition 3.25 we have that a permutation σ is odd if and only if the number of cycles of even length in a cycle decomposition is odd. If σ has an odd number of cycles of even length in a decomposition, then the composition σ^2 must have an even number of cycles of even length, and therefore be even.

Question 2

Use the class equation to find all finite groups which have exactly two conjugacy classes.

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Solution. The group Z_2 has two singleton conjugacy classes, since $Z(Z_2) = Z_2$. Suppose that G is a finite group other than Z_2 which has exactly two conjugacy classes. One of those classes must be $\{1\}$, and therefore the other class must be a non-singleton class. Suppose this class has representative g. By the class equation (Thm 4.7), this gives

$$|G| = 1 + |G : C_G(g)|.$$

By Lagrange's Theorem (3.8), must have that $|G : C_G(g)| = |G| - 1$ divides $|G|$. This is only possible if $|G| = 2$, but that implies that $G = Z_2$. Therefore, Z_2 is the only group with exactly two conjugacy classes.

Question 3

Prove that if $P \in Syl_p(G)$ and H is a subgroup of G containing P, then $P \in Syl_p(H)$.

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Solution. If $P \in Syl_p(G)$ and $P \leq H$, then $|G| = p^\alpha m$, where p does not divide m, and $|H| = p^\alpha k$. By Lagrange, we have that $|H|$ divides $|G|$ and therefore k divides m. Therefore, p cannot divide k. This implies that $P \in Syl_p(H)$.

Question 4

Show that the center of a direct product is the direct product of the centers:

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n).$$

Deduce that a direct product of groups is abelian if and only if each of the factors is abelian.

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Solution. Note that for any $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in G_1 \times G_2 \times \cdots \times G_n$, we have that

$$(a_1, \ldots, a_n)(b_1, \ldots, b_n) = (b_1, \ldots, b_n)(a_1, \ldots, a_n) \Leftrightarrow \ a_i b_i = b_i a_i, \quad 1 \leq i \leq n,$$

by the definition of direct products. Therefore an element $(g_1, \ldots, g_n)$ of $G_1 \times G_2 \times \cdots \times G_n$ is in the center if and only if every g_i is in $Z(G_i)$. This implies that

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n).$$

If the direct product is abelian, then

$$Z(G_1 \times G_2 \times \cdots \times G_n) = G_1 \times G_2 \times \cdots \times G_n = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n),$$

which implies that $G_i = Z(G_i)$ for $1 \leq i \leq n$. Conversely, if $G_i = Z(G_i)$ for $1 \leq i \leq n$, then

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n) = G_1 \times G_2 \times \cdots \times G_n,$$

which implies that the direct product is abelian.

Question 5

In each part, give the list of invariant factors for all abelian groups of the specified order:

(a) 270,

(b) 9801.

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Solution.

(a) The prime factorization of 270 is $2 \cdot 3^3 \cdot 5$. Therefore, we must have that n_1 is divisible by 2,3,5. This leaves three options based on the three integer partitions of 3: $3^3, 3^2 \cdot 3^1, 3^1 \cdot 3^1 \cdot 3^1$:

n_1	n_2	n_3
$2 \cdot 3^3 \cdot 5$		
$2 \cdot 3^2 \cdot 5$	3	
$2 \cdot 3 \cdot 5$	3	3

(b) The prime factorization of 9801 is $3^4 \cdot 11^2$. Therefore, we must have that n_1 is divisible by 3 aand 11. We have 5 choices for breaking up the powers of 3, based on the integer partitions of 4: 3^4, $3^3 \cdot 3^1$, $3^2 \cdot 3^2$, $3^2 \cdot 3^1 \cdot 3^1$, $3^1 \cdot 3^1 \cdot 3^1 \cdot 3^1$. We have 2 choices for breaking up the powers of 11: 11^2 and $11^1 \cdot 11^1$. This gives a total of 10 possible lists of invariant factors:

n_1	n_2	n_3	n_4
$3^4 \cdot 11^2$			
$3^3 \cdot 11^2$	3		
$3^2 \cdot 11^2$	3^2		
$3^2 \cdot 11^2$	3	3	
$3 \cdot 11^2$	3	3	3
$3^4 \cdot 11$	11		
$3^3 \cdot 11$	$3 \cdot 11$		
$3^2 \cdot 11$	$3^2 \cdot 11$		
$3^2 \cdot 11$	$3 \cdot 11$	3	
$3 \cdot 11$	$3 \cdot 11$	3	3