124 Linear Algebra - Midterm 2 - Solutions

Question 1

Consider the map $h : \mathcal{P}_2 \to \mathcal{P}_2$ given by $h(a + bx + cx^2) = (a + b) + (a + b)x + cx^2$.

(a) Show that this map is a homomorphism.

(b) Find the range space and the null space of this map.

(c) Let $B = \langle 1, x, x + x^2 \rangle$. Find $Rep_B(1 + x + x^2)$ and $Rep_B(h(1 + x + x^2))$.

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Solution.

(a) We show that this map preserves lineara combinations of pairs of polynomials:

$$\begin{aligned} h(r(a + bx + cx^2) + r'(a' + b'x + c'x^2)) &= h((ra + r'a') + (rb + r'b')x + (rc + r'c')x^2) \\ &= ((ra + r'a') + (rb + r'b')) + ((ra + r'a') + (rb + r'b'))x + (rc + r'c')x^2 \\ &= r \cdot ((a + b) + (a + b)x + cx^2) + r' \cdot ((a' + b') + (a' + b')x + c'x^2) \\ &= r \cdot h(a + bx + cx^2) + r' \cdot h(a' + b'x + c'x^2). \end{aligned}$$

(b) Since a, b, c can be any real numbers, we see thaat $a + b$ and c can both be any real numbers. Therefore, the range space is

$$\mathcal{R}(h) = \{(a + b) + (a + b)x + cx^2 \mid a, b, c \in \mathbb{R}\} = \{\alpha + \alpha x + cx^2 \mid \alpha, c \in \mathbb{R}\}.$$

Therefore, $\mathcal{R}(h)$ has a basis $\langle 1 + x, x^2 \rangle$. The null space is

$$\mathcal{R}(h) = \{a + bx + cx^2 \mid a + b = 0, c = 0,\ a, b, c \in \mathbb{R}\} = \{a - ax \mid a, c \in \mathbb{R}\}.$$

Therefore, $\mathcal{N}(h)$ has a basis $\langle 1 - x \rangle$.

(c) We have

$$1 + x + x^2 = 1(1) + 1(x + x^2) \Rightarrow Rep_B(1 + x + x^2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_B,$$

and

$$h(1 + x + x^2) = 2 + 2x + x^2 = 2(1) + 1(x) + 1(x + x^2) \Rightarrow Rep_B(h(1 + x + x^2)) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}_B.$$

Question 2

Assume h is a linear transformation of V and that $\langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ is a basis of V. Prove the following statements.

(a) If $h(\vec{\beta}_i) = \vec{0}$ for each basis vector then h is the zero map.

(b) If $h(\vec{\beta}_i) = \vec{\beta}_i$ for each basis vector then h is the identity map.

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Solution.

(a) For any $\vec{v} \in V$, we have $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Then

$$h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) = c_1\vec{0} + \cdots + c_n\vec{0} = \vec{0}.$$

Therefore, h is the zero map.

(b) For any $\vec{v} \in V$, we have $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Then

$$h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n = \vec{v}.$$

Therefore, h is the identity map.

Question 3

Consider reflection through the x-axis in $\mathbb{R}^2$. This is the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} x \\ -y \end{smallmatrix}\right)$.

(a) Find the matrix that represents this map, with respect to the standard basis $\langle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \rangle$.

(b) Find the matrix that represents this map, with respect to the basis $B = \langle \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \rangle$.

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Solution.

(a) We see that the map sends $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$. Since

$$Rep_E\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)_E, \text{ and } Rep_E\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)_E,$$

we have that

$$Rep_{E,E} f = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)_{E,E}.$$

(b) We see that the map sends $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Since

$$Rep_E\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)_E, \text{ and } Rep_E\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)_E,$$

we have that

$$Rep_{E,E} f = \left(\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right)_{E,E}.$$

Question 4

Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find the dimension of the null space and the range space of a map represented by this matrix.

(b) Suppose that this matrix represents a linear map $f : \mathcal{P}_2 \to \mathbb{R}^2$ with respect to their standard bases. What is $f(x)$? What is $f^{-1}(\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))$?

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Solution.

(a) The matrix is already in rref, so we see immediately that the range space (the span of the columns) is 2-dimensional. Since the domain is 3-dimensional (A is a 2×3 matrix), we have by the rank-nullity theorem that the null space is 1-dimensional.

(b) Let $B = \langle 1, x, x^2 \rangle$. We have $Rep_B x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_B$. This gives

$$f(x) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{B,E} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The preimage $f^{-1}(\begin{pmatrix} 0 \\ 1 \end{pmatrix})$ is the set that sends $a + bx + cx^2$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ via this map defined by A. This is the set

$$f^{-1}(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \{a + bx + cx^2 \mid a + b = 0, c = 1\} = \{a(1 - x) + x^2 \mid a \in \mathbb{R}\}.$$

Question 5

Let V, W be vector spaces and consider a homomorphism $f : V \to W$.

(a) Show that the null space $\mathcal{N}(f)$ is a subspace of V.

(b) Use the rank nullity theorem to show that the rank of f is at most the dimension of V.

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Solution.

(a) We have that

$$\mathcal{N}(f) = \{\vec{v} \in V \mid f(\vec{v}) = \vec{0}\}.$$

Suppose that $\vec{v}_1, \vec{v}_2 \in \mathcal{N}(f)$ and $r_1, r_2 \in \mathbb{R}$, then

$$f(r_1\vec{v}_1 + r_2\vec{v}_2) = r_1 f(\vec{v}_1) + r_2 f(\vec{v}_2) = r_1\vec{0} + r_2\vec{0} = \vec{0},$$

since homomorphisms preserve linear combinations. Therefore $\mathcal{N}(f)$ is closed under taking linear combinations and is therefore a subspace.

(b) By the rank nullity theorem, we have $\dim V = \text{rank} f + \text{null} f$. Since $\text{null} f = \dim \mathcal{N}(f)$ it is a nonnegative integer, which implies that $\dim V \geq \text{rank} f$.