124 Linear Algebra - Midterm 2 - Practice - Solutions

Question 1

Consider the map $h : \mathcal{P}_2 \to \mathcal{P}_2$ given by $h(a + bx + cx^2) = bx^2 - (a + c)x + a$.

(a) Show that this map is a homomorphism.

(b) Find the range space and the null space of this map.

(c) Let $B = \langle 1, x, x + x^2 \rangle$. Find $Rep_B(1 + x^2)$ and $Rep_B(h(1 + x^2))$.

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Solution.

(a) In order to show that this is a homomorphism, we show that this map preserves linear combinations of two vectors. Let $a + bx + cx^2, a' + b'x + c'x^2 \in \mathcal{P}_2$ and $r, r' \in \mathbb{R}$. Then

$$\begin{aligned} r(a + bx + cx^2) + r'(a' + b'x + c'x^2) &= (ra + r'a') + (rb + r'b')x + (rc + r'c')x^2) \\ &\mapsto (ra + r'a') - ((ra + r'a') + (rc + r'c'))x + (rb + r'b')x^2 \\ &= r(a - (a + c)x + bx^2) + r'(a' - (a' + c')x + b'x^2). \end{aligned}$$

(b) We have that $h(a + bx + cx^2) = a + (a + c)x + bx^2 = 0$ implies that $a = 0, a + c = 0, b = 0$, which is easily seen to imply that $a = b = c = 0$. Therefore, the null space of this map is the trivial space $\{0\}$. By the fact that $\dim \mathcal{P}_2 = 3$ and the rank nullity theorem, we see that the range space must then be 3-dimensional. It is also a subspace of the 3-dimensional space $\mathcal{P}_2$, and therefore the range space is $\mathcal{P}_2$. (We could also have started by showing the range space is $\mathcal{P}_2$, and then used rank-nullity to conclude that the null space is $\{0\}$, or found them both separately without using this theorem.)

(c) We can write

$$1 + x^2 = 1 - x + x + x^2,$$

and therefore

$$Rep_B(1 + x^2) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}_B.$$

Similarly, we write

$$h(1 + x^2) = 1 - 2x,$$

and therefore

$$Rep_B(h(1 + x^2)) = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}_B.$$

Question 2

Let $f : V \to W$ be a homomorphism. Show that f preserves linear dependence. In other words, if a set $(\vec{v}_1, \ldots, \vec{v}_k)$ is linearly dependent in V, then $(f(\vec{v}_1), \ldots, f(\vec{v}_k))$ is linearly dependent in W.

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Solution. Suppose that the vectors $(\vec{v}_1, \ldots, \vec{v}_k)$ form a linearly dependent set in V. Then there exists a relation

$$a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = \vec{0}$$

such that not $a_1 = \cdots = a_k = 0$. Since $f(\vec{0}) = \vec{0}$ always, we have

$$f(\vec{0}) = f(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k) = a_1 f(\vec{v}_1) + \cdots + a_k f(\vec{v}_k) = \vec{0},$$

since homomorphisms preserve linear combinations. This implies that we have a nontrivial relation on the set $(f(\vec{v}_1), \ldots, f(\vec{v}_k))$ which is therefore linearly dependent.

Question 3

Let $B = \langle\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3, \vec{\beta}_4\rangle$ be a basis for a vector space V. Find a matrix with respect to B, B for each of the following transformations of V determined by:

(a) $\vec{\beta}_1 \mapsto \vec{\beta}_2,\ \vec{\beta}_2 \mapsto \vec{\beta}_3,\ \vec{\beta}_3 \mapsto \vec{\beta}_4, \vec{\beta}_4 \mapsto \vec{0}$.

(b) $\vec{\beta}_1 \mapsto \vec{\beta}_2,\ \vec{\beta}_2 \mapsto \vec{0},\ \vec{\beta}_3 \mapsto \vec{\beta}_4, \vec{\beta}_4 \mapsto \vec{\beta}_1$.

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Solution. In each case, let $f : V \to V$.

(a) We have:

$$Rep_B(f(\vec{\beta}_1)) = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}_B,\quad Rep_B(f(\vec{\beta}_2)) = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}_B,\quad Rep_B(f(\vec{\beta}_3)) = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}_B,\quad Rep_B(f(\vec{\beta}_4)) = \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}_B.$$

And therefore

$$Rep_{B,B}(f) = \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0 \end{pmatrix}_{B,B}.$$

(b) We have:

$$Rep_B(f(\vec{\beta}_1)) = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}_B,\quad Rep_B(f(\vec{\beta}_2)) = \begin{pmatrix} 0\\0\\0\\0 \end{pmatrix}_B,\quad Rep_B(f(\vec{\beta}_3)) = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}_B,\quad Rep_B(f(\vec{\beta}_4)) = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}_B.$$

And therefore

$$Rep_{B,B}(f) = \begin{pmatrix} 0&0&0&1\\1&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}_{B,B}.$$

Question 4

Consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & -3\\ 2 & 5 & 0 \end{pmatrix}.$$

(a) Find the dimension of the domain and the codomain of an map represented by this matrix.

(b) Give an explicit example of a domain with appropriate basis, a codomain with appropriate basis, and a map such that the matrix A represents the map with respect to your chosen bases.

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Solution.

(a) This map takes as input vectors of length 3 and outputs vectors of length 2, since this is a 2×3 matrix. Therefore the domain is 3-dimensional and the codomain is 2-dimensional.

(b) Let's consider a map $f : \mathbb{R}^3 \to \mathbb{R}^2$, with the standard basis on both. Then we can let f simply be defined by

$$\vec{v} \mapsto \begin{pmatrix} 1 & 1 & -3 \\ 2 & 5 & 0 \end{pmatrix} \vec{v}.$$

In other words, this is the map that maps the standard basis as follows:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Question 5

Let V, W, U be vector spaces and consider two homomorphisms $f : V \to W$ and $h : W \to U$.

(a) Use the rank nullity theorem to show that the rank of $h \circ f$ is at most the minimum of the ranks of h and f.

(b) To show that the rank of $h \circ f$ can be less than that, give an example of two 2×2 matrices A, B that both have rank 1, such that AB has rank 0.

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Solution.

(a) The range space $\mathcal{R}(h \circ f)$ is a subspace of $\mathcal{R}(h)$, since

$$\mathcal{R}(h \circ f) = \{h(f(\vec{v})) \in U \mid \vec{v} \in V\} = \{h(\vec{w}) \in U \mid \vec{w} \in \mathcal{R}(f)\} \subseteq \{h(\vec{w}) \in U \mid \vec{w} \in W\}.$$

Therefore, the rank of $h \circ f$ is at most the rank of h (since rank is the dimension of the range space).

Since $\mathcal{R}(h \circ f)$ is the image of $\mathcal{R}(f)$ mapped by h, we also have, by the rank nullity theorem, that the dimension of $\mathcal{R}(h \circ f)$ is bounded by the dimension of $\mathcal{R}(f)$. Therefore, the rank of $h \circ f$ is at most the rank of f.

Putting those two together, we see that the rank of $h \circ f$ is at most the minimum of the ranks of h and f.

(b) For example, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

such that

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now if A describes a linear map h and B describes a linear map f, then h and f both have rank 1, but $h \circ f$ is the zero map which has rank 0.