124 Linear Algebra - Midterm 1 - Solutions

Question 1

Let

$$S = \left\{ \begin{pmatrix} a \\ b \\ a+b \\ a-b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

(a) Show that S is closed under vector addition.

(b) You may assume that S is a subspace. Find a basis B for S.

(c) Use your basis B to write the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix} \in S$ as represented by B, i.e. find the vector

$$\mathrm{Rep}_B\,(\vec{v}$$

Hint: your answer should be a vector of length 2, but the entries will depend on your choice of basis in (b).

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Solution.

(a) Let $\vec{v}$ and $\vec{w}$ be two vectors in S. Then we can write:

$$\vec{v} + \vec{w} = \begin{pmatrix} a \\ b \\ a+b \\ a-b \end{pmatrix} + \begin{pmatrix} c \\ d \\ c+d \\ c-d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ a+b+c+d \\ a-b+c-d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ (a+c)+(b+d) \\ (a+c)-(b+d) \end{pmatrix} \in S.$$

Therefore, S is indeed closed under vector addition.

(b) We rewrite S as

$$S = \left\{ \begin{pmatrix} a \\ b \\ a+b \\ a-b \end{pmatrix} \mid a, b \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} a \\ 0 \\ a \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ b \\ b \\ -b \end{pmatrix} \mid a, b \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

So, we see that S can be expressed as the span of $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. To show that these two vectors are linearly independent, we check that

$$RREF\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which indeed has 2 leading 1s. Since this pair of vectors both spans S and is linearly independent, it is a basis for S, and we can let

$$B = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right).$$

(c) In this case, it is not hard to spot the linear combination, but to solve this systematically, we find

$$RREF\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From this, we see that

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix},$$

and therefore

$$\mathrm{Rep}_B\,(\vec{v}) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_B.$$

Question 2

Consider the following subset of $\mathcal{P}_3$:

$$S = \{1 + x, 1 + x^2, 1 + x^3\}.$$

(a) Write each element of S in terms of the basis $E = (1, x, x^2, x^3)$. Then, show that this set S is linearly independent.

(b) We know that $\mathcal{P}_3$ is 4-dimensional, and therefore S does not span all of $\mathcal{P}_3$. Find an element in $\mathcal{P}_3$ that is not in the span of S. For this part, you do not need to justify your answer.

(c) Give examples of a 1-dimensional, 2-dimensional and 3-dimensional subspace of $\mathcal{P}_3$, respectively.

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Solution.

(a) We see that

$$\text{Rep}_E(1 + x) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}_E$$
$$\text{Rep}_E\left(1 + x^2\right) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_E$$
$$\text{Rep}_E\left(1 + x^3\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}_E$$

To show that this set is linearly independent, we see that

$$RREF\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(The fastest way to get this RREF is to first move the first row to the end, then subtract rows 1-3 once from the last one.)

(b) For example, the element 1 is not in the span of S.

(c) We can start from the standard basis $E = (1, x, x^2, x^3)$, and consider the subspaces $[(1)]$, $[(1, x)]$, $[(1, x, x^2)]$, which have dimension 1, 2, 3 respectively. (This follows from the linear independence of E.) These are in fact the spaces $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$.

Question 3

Consider a pair of vectors $\vec{u}, \vec{v}$ in a vector space V.

(a) Show that the vector $\vec{u}$ is in the span of the set $(\vec{u} + \vec{v}, \vec{v})$.

(b) Suppose that the set $(\vec{u}, \vec{v})$ is linearly independent. What is the dimension of the subspace $[(\vec{u}, \vec{v}, \vec{u} + \vec{v})]$?

(c) Show carefully that if two vectors $\vec{x}$ and $\vec{y}$ are both linear combinations of $\vec{u}$ and $\vec{v}$, then the vector $\vec{x} + \vec{y}$ is also a linear combination of $\vec{u}$ and $\vec{v}$.

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Solution.

(a) We can write

$$\vec{u} = 1 \cdot (\vec{u} + \vec{v}) + (-1) \cdot \vec{v},$$

and therefore $\vec{u} \in [(\vec{u} + \vec{v}, \vec{v})]$.

(b) It is easy to see that $\vec{u} + \vec{v}$ is in the span of $(\vec{u}, \vec{v})$, and therefore $(\vec{u}, \vec{v})$ is linearly independent and spans the subspace $[(\vec{u}, \vec{v}, \vec{u} + \vec{v})]$. Therefore, this subspace has dimension 2.

(c) We have

$$\begin{aligned} \vec{x} &= a\vec{u} + b\vec{v} \\ \vec{y} &= c\vec{u} + d\vec{v} \\ \vec{x} + \vec{y} &= a\vec{u} + b\vec{v} + c\vec{u} + d\vec{v} = (a + c)\vec{u} + (b + d)\vec{v}. \end{aligned}$$

Therefore, $\vec{x} + \vec{y}$ is also a linear combination of $\vec{u}$ and $\vec{v}$.

Question 4

Suppose that W is a subspace of $\mathbb{R}^2$.

(a) Show that $\dim W \leq 2$.

(b) Suppose that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin W$. What does this tell us about $\dim W$?

(c) Give an example of a subspace U of $\mathbb{R}^2$ such that there is only one element that is both in W and in U.

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Solution.

(a) Let B be a basis for the subspace W. Then the set B is linearly independent in $\mathbb{R}^2$ and can therefore be extended to a basis of $\mathbb{R}^2$. Since bases of $\mathbb{R}^2$ have cardinality 2, we must have $|B| \leq 2$ and therefore $\dim W \leq 2$.

(b) Since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W$, we know that $\dim W \geq 1$, since we know that the set $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ can be extended to a basis of W. However, if W was 2-dimensional, we would have a basis for W that was a linearly independent set of size 2 in $\mathbb{R}^2$, and therefore a basis for $\mathbb{R}^2$, implying that $W = \mathbb{R}^2$. Since this is not the case, we must have $\dim W = 1$.

(c) We can let $U = \{\vec{0}\}$, the trivial subspace. In this case, there is only one element in U, and it must also be in W since every subspace contains $\vec{0}$.

Question 5

We would like to solve the following system of equations:

$$\begin{array}{cccccc} 2x & & +2z & = & 4 \\ & y & & = & 3 \\ 2x & +y & +3z & = & 8. \end{array}$$

(a) Find an explicit matrix A and vector $\vec{w}$ and express the system of equations in the form

$$A\vec{x} = \vec{w},$$

where $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(b) Find the augmented matrix of this system, and then write your augmented matrix in RREF. [4 points]

(c) What is the set of solutions to this system?

(d) What is the rank of A?

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Solution.

(a) This system is equivalent to solving the matrix equation

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}.$$

(b) We write this as the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 0 & 2 & 4 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & 3 & 8 \end{array}\right).$$

To write this in RREF, we perform the following elementary row operations: multiply r_1 by $\frac{1}{2}$, subtract r_1 once from r_2 and 2 times from r_3, and then subtract r_2 once from r_3, to obtain:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

(c) From the rref of the augmented matrix, we read that there is a unique solution to this system, given by

$$x = 1$$
$$y = 3$$
$$z = 1.$$

(d) The matrix A has 3 leading 1s in its rref, and therefore has rank 3.