124 Linear Algebra - Midterm 1 - Solutions

Question 1

Let

$$S = \left\{ \begin{pmatrix} a \\ b \\ c \\ a+b+c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

(a) Show that S is closed under scalar multiplication.

(b) You may assume that S is a subspace. Find a basis B for S.

(c) Use your basis B to write the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix} \in S$ as represented by B, i.e. find the vector

$$\mathrm{Rep}_B(\vec{v}).$$

Hint: your answer should be a vector of length 3, but the entries will depend on your choice of basis in (b).

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Solution.

(a) Let $\vec{v} = \begin{pmatrix} a \\ b \\ c \\ a+b+c \end{pmatrix} \in S$ and let $r \in \mathbb{R}$. Then

$$r \cdot \vec{v} = r \cdot \begin{pmatrix} a \\ b \\ c \\ a+b+c \end{pmatrix} = \begin{pmatrix} ra \\ rb \\ rc \\ r(a+b+c) \end{pmatrix} = \begin{pmatrix} ra \\ rb \\ rc \\ ra+rb+rc \end{pmatrix} \in S.$$

(b) We can rewrite

$$S = \left\{ \begin{pmatrix} a \\ b \\ c \\ a+b+c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\},$$

and see that S is the span of the set

$$B = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right).$$

To see that this set is indeed inpendent, we can note that each vector in B has a 1 in a position where no other vector in B has a nonzero entry, and therefore no vector in B can be a linear combination of another. Alternatively, we can also write

$$\mathrm{RREF}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and see that each column in the RREF has a leading 1, and therefore the set of columns in the original matrix are independent.

(c) We see that

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

and therefore

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_B.$$

Question 2

Let V be a vector space. Show (carefully and concisely) that V has exactly one subspace on a finite number of elements.

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Solution. Every vector space V contains the trivial space $\{\vec{z}\}$ as a subspace. Therefore, V has at least one subspace on a finite number of elements. To show that there cannot be more, suppose that $S = \{\vec{v}_1, \ldots, \vec{v}_n\} \neq \{\vec{z}\}$ is a different subspace of V on a finite number of elements. Then S must contain at least one vector that is not $\vec{z}$. (Otherwise, $S = \{\vec{z}\}$.) Let $\vec{v} \in S$ with $\vec{v} \neq \vec{z}$. Then for any $r \in \mathbb{R}$, we must have $r\vec{v} \in S$, and since these are all distinct, we now have infinitely many elements in S.

Comment: In the above argument, we did not prove carefully that $r_2\vec{v}$ is distinct from $r_2\vec{v}$ when $r_1 \neq r_2$. This follows from elementary properties of vectors spaces. During an exam, you may always ask me if you are allowed to assume certain facts or need to show them.

Question 3

Consider a pair of vectors $\vec{u}, \vec{v}$ in a vector space V. We will show that

$$[(\vec{u}, \vec{v})] = [\vec{u} + \vec{v}, \vec{u} - \vec{v}],$$

by showing that $[(\vec{u}, \vec{v})] \subseteq [\vec{u} + \vec{v}, \vec{u} - \vec{v}]$ and then that $[\vec{u} + \vec{v}, \vec{u} - \vec{v}] \subseteq [(\vec{u}, \vec{v})]$.

(a) Show that $[(\vec{u}, \vec{v})] \subseteq [\vec{u} + \vec{v}, \vec{u} - \vec{v}]$. Note that spans are subspaces, which are therefore closed under taking linear combinations. Therefore, to show that $[(\vec{u}, \vec{v})] \subseteq W$, you only need to show $\vec{u}, \vec{v} \in W$.

(b) Show that $[\vec{u} + \vec{v}, \vec{u} - \vec{v}] \subseteq [(\vec{u}, \vec{v})]$.

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Solution.

(a) The hint helps us simplify this question. In order to show that $[(\vec{u}, \vec{v})] \subseteq [\vec{u} + \vec{v}, \vec{u} - \vec{v}]$, we only need to show that $\vec{u}$ and $\vec{v}$ can both be written as a linear combination of $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$. We see that

$$\begin{aligned} \vec{u} &= \tfrac{1}{2} \cdot (\vec{u} + \vec{v}) + \tfrac{1}{2} \cdot (\vec{u} - \vec{v}) \\ \vec{v} &= \tfrac{1}{2} \cdot (\vec{u} + \vec{v}) - \tfrac{1}{2} \cdot (\vec{u} - \vec{v}). \end{aligned}$$

(b) Similarly, we only need to show that $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are in the span of $\vec{u}$ and $\vec{v}$. We see that quite easily:

$$\begin{aligned} \vec{u} + \vec{v} &= 1 \cdot \vec{u} + 1 \cdot \vec{v} \\ \vec{u} - \vec{v} &= 1 \cdot \vec{u} + (-1) \cdot \vec{v}. \end{aligned}$$

Question 4

Find a basis B for the vector space

$$\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$$

such that your basis includes the element $1+x$. You may assume that $\dim \mathcal{P}_2$=3. Therefore, it is sufficient to find a set of 3 elements and show that it is a linearly independent set.

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Solution. We propose the basis $B = (1, 1+x, x^2)$. There are different ways of arguing that this set is linearly independent. One way is to write it in terms of the standard basis $E = (1, x, x^2)$ as $B = \left(\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)_E, \left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)_E, \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)_E\right)$. Taking

$$\text{RREF}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

which shows that these columns are indeed independent.

Question 5

We would like to solve the following system of equations:

$$\begin{array}{lllll} 2x & & & = & 6 \\ 2x & +y & +z & = & 5 \\ 4x & +y & +z & = & 11. \end{array}$$

(a) Find an explicit matrix A and vector $\vec{w}$ and express the system of equations in the form

$$A\vec{x} = \vec{w},$$

where $\vec{x} = \left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)$.

(b) Find the augmented matrix of this system, and then write your augmented matrix in RREF.

(c) What is the set of solutions to this system?

(d) What is the rank of A?

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Solution.

(a) This system is equivalent to solving the matrix equation

$$\begin{pmatrix}2&0&0\\2&1&1\\4&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}6\\5\\11\end{pmatrix}.$$

(b) We write this as the augmented matrix:

$$\left(\begin{array}{ccc|c}2&0&0&6\\2&1&1&5\\4&1&1&11\end{array}\right).$$

To write this in RREF, we perform the following elementary row operations: multiply r_1 by $\frac{1}{2}$, subtract r_1 two times from r_2 and 4 times from r_3, and then subtract r_2 once from r_3, to obtain:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

(c) By the augmented matrix, we see that the original system of equations is equivalent to the system:

$$\begin{array}{ccccc} x & & & = & 3 \\ & y & +z & = & -1. \end{array}$$

We have $x = 3$. We can let $z = t$ for any $t \in \mathbb{R}$, and then $y = -t - 1$. This gives an infinte set of solutions to this system of equations.

(d) We have seen that the rank of A is the number of leading 1s in the RREF of A. So, the rank is 2.