124 Linear Algebra - Midterm 1

Name:

Justify all of your answers.

Question 1

Let

$$S = \left\{ \begin{pmatrix} a \\ b \\ a+b \\ a-b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

(a) Show that S is closed under vector addition. [3 points]

(b) You may assume that S is a subspace. Find a basis B for S. [4 points]

(c) Use your basis B to write the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix} \in S$ as represented by B, i.e. find the vector [3 points]

$$\mathrm{Rep}_B(\vec{v}).$$

Hint: your answer should be a vector of length 2, but the entries will depend on your choice of basis in (b).

•••••••••

Question 2

Consider the following subset of $\mathcal{P}_3$:

$$S = \{1 + x, 1 + x^2, 1 + x^3\}.$$

(a) Write each element of S in terms of the basis $E = (1, x, x^2, x^3)$. Then, show that this set S is linearly independent. [5 points]

(b) We know that $\mathcal{P}_3$ is 4-dimensional, and therefore S does not span all of $\mathcal{P}_3$. Find an element in $\mathcal{P}_3$ that is not in the span of S. For this part, you do not need to justify your answer. [2 points]

(c) Give examples of a 1-dimensional, 2-dimensional and 3-dimensional subspace of $\mathcal{P}_3$, respectively. [3 points]

....

Question 3

Consider a pair of vectors $\vec{u}, \vec{v}$ in a vector space V.

(a) Show that the vector $\vec{u}$ is in the span of the set $(\vec{u} + \vec{v}, \vec{v})$. [4 points]

(b) Suppose that the set $(\vec{u}, \vec{v})$ is linearly independent. What is the dimension of the subspace $[(\vec{u}, \vec{v}, \vec{u} + \vec{v})]$? [3 points]

(c) Show carefully that if two vectors $\vec{x}$ and $\vec{y}$ are both linear combinations of $\vec{u}$ and $\vec{v}$, then the vector $\vec{x} + \vec{y}$ is also a linear combination of $\vec{u}$ and $\vec{v}$. [3 points]

....

Question 4

Suppose that W is a subspace of $\mathbb{R}^2$.

(a) Show that $\dim W \leq 2$. [4 points]

(b) Suppose that $\binom{1}{0} \in W$ and $\binom{1}{1} \notin W$. What does this tell us about $\dim W$? [3 points]

(c) Give an example of a subspace U of $\mathbb{R}^2$ such that there is only one element that is both in W and in U. [3 points]

....

Question 5

We would like to solve the following system of equations:

$$\begin{array}{rrrcl} 2x & & +2z & = & 4 \\ & y & & = & 3 \\ 2x & +y & +3z & = & 8. \end{array}$$

(a) Find an explicit matrix A and vector $\vec{w}$ and express the system of equations in the form [2 points]

$$A\vec{x} = \vec{w},$$

where $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(b) Find the augmented matrix of this system, and then write your augmented matrix in RREF. [4 points]

(c) What is the set of solutions to this system? [2 points]

(d) What is the rank of A? [2 points]

....

Question 6

Let U and W both be subspaces of $\mathbb{R}^3$, with $\dim U = \dim W = 2$. Prove that $\dim U \cap W \geq 1$. **[0 points]**

In general, show that if U, W are subspaces of a vector space V, then $\dim U \cap W \geq \dim U + \dim W - \dim V$.

....