124 Linear Algebra - Final - Practice Solutions

Question 1

(a) **Without justification**, cross out the items in the list which do not apply:

For **every** square matrix A, if we know the characteristic polynomial of a A, then we can infer the

- eigenvalues
- ~~eigenvectors~~
- ~~eigenspaces~~
- trace
- determinant
- algebraic multiplicities
- ~~geometric multiplicities~~
- invertibility
- ~~rank~~

of A.

(b) *[6 points]* Find all the eigenvalues and eigenspaces of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 4 & 2 \\ 8 & 0 & 5 \end{pmatrix}.$$

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Solution. First, we find the characteristic polynomial of A:

$$f_A(\lambda) = -\lambda^3 + 10\lambda^2 - 29\lambda + 20 = (5-\lambda)(4-\lambda)(1-\lambda).$$

We see that A has eigenvalues $\lambda_1 = 5$, $\lambda_2 = 4$, $\lambda_3 = 1$. For the eigenspacess, we have

$$E_5 = \ker \begin{pmatrix} -4 & 0 & 0 \\ 4 & -1 & 2 \\ 8 & 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right],$$

$$E_4 = \ker \begin{pmatrix} -3 & 0 & 0 \\ 4 & 0 & 2 \\ 8 & 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right],$$

$$E_4 = \ker \begin{pmatrix} 0 & 0 & 0 \\ 4 & 3 & 2 \\ 8 & 0 & 4 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right].$$

Question 2

Let $L \subset \mathbb{R}^2$ be the span of $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

(a) *[4 points]* Give the 2×2 matrix A of the projection onto L, and find the projections of the standard basis vectors. *Hint: use Gram-Schmidt.*

(b) *[6 points]* Is A diagonalizable? If so, give an invertible matrix S and a diagonal matrix B, such that $A = SBS^{-1}$. Otherwise, explain why A is not diagonalizable.

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Solution.

(a) Using Gram-Schmidt, we find an orthonormal basis for L as $\begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$ (since this is only one vector all we have to do is make it unit length). We let $Q = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$, and then we find the matrix

$$A = QQ^T = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}\begin{pmatrix} -3/5 & 4/5 \end{pmatrix} = \begin{pmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{pmatrix}.$$

Then, the projections on the standard basis vectors onto L are given by

$$\begin{pmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9/25 \\ -12/25 \end{pmatrix} \text{ and } \begin{pmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -12/25 \\ 16/25 \end{pmatrix}.$$

(b) It is easy to see that A hhas ann eigenbasis, since it has 2 distinct eigenvectors $\lambda_1 = 1$ and $\lambda_2 = 0$, with eigenvectors $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, respectively. Therefore, A is diagonalizable as

$$A = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}^{-1}.$$

Question 3

(a) *[6 points]* Diagonalize the following matrix:

$$A = \begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}$$

(b) *[4 points]* Can you diagonalize A^2? In general, what do the powers $A^2, A^3, \dots$ of a diagonalizable matrix look like? *Hint: do not try to square A directly. Instead notice that* $SBS^{-1}SBS^{-1} = SB^2S^{-1}$.

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Solution.

(a) A hhas characteristic polyomial $f_A(\lambda) = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1)$ and eigenvalues s$\lambda_1 = 5$ and $\lambda_2 = 1$. We find the eigenspaces

$$E_5 = \ker\begin{pmatrix} 0 & 4 \\ 0 & -4 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right],$$

$$E_1 = \ker\begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right].$$

Therefore, we can diagonalize A as

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 5 & - \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1}.$$

(b) For any power A^k, we have

$$A^k = (SBS^{-1})^k = \underbrace{SBS^{-1}SBS^{-1}\dots SBS^{-1}SBS^{-1}}_{k \text{ times}}$$
$$= SB(S^{-1}S)B(S^{-1}S)\dots(S^{-1}S)B(S^{-1}S)BS^{-1} = SB^kS^{-1}.$$

Furthermore, we see that for any diagonal matrix

$$B^k = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix},$$

which is also diagonal. Therefore, if A is diagonalizable, then so is A^k, and its eigenvalues are simply the eigenvalues of A raised to the power k. Note that the eigenvectors of A^k are the same as those of A.

Question 4

(a) *[6 points]* Show that the composition of linear transformations on $\mathbb{R}^1$ is commutative. *Hint: you have shown before that linear transformations on $\mathbb{R}^1$ have a very restricted form.*

(b) *[4 points]* Is this true for linear transformations on $\mathbb{R}^n$ in general? Prove or give a counter-example.

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Solution.

(a) As we have shown in Exercise 5 on HW5, a linear transformation $f : \mathbb{R} \to \mathbb{R}$ takes the form $f(x) = cx$, for c a scalar. Therefore if we have two transformations $f(x) = c_1x$ and $g(x) = c_2x$, then the compositions

$$f(g(x)) = c_1(c_2x) = c_1c_2x = c_2c_1x = g(f(x)),$$

always commute.

(b) This is not true in general. For example, take two linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ given by the matrices $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, then the compositions are given by AB or BA. However these matrices do not commute:

$$AB = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$$

and

$$BA = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Question 5

Show that λ is an eigenvalue of an $n \times n$ matrix A if and only if the map represented by $A - \lambda I_n$ is not an isomorphism. *Hint: what do you know about the kernel of isomorphisms?*

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Solution. We have that λ is an eigenvalue of A with nonzero eigenvector $\vec{v}$ if and only if $A\vec{v} = \lambda\vec{v}$, which we can rewrite as $(A - \lambda I_n)\vec{v} = \vec{0}$. Therefore, λ is an eigenvalue of A if and only if the matrix $A - \lambda I_n$ has nonzero kernel. We have seen that a linenar map $\mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism if and only if it is injective. Furthermore, a linear map is injective if and only if it has zero kernel. Therefore, λ is an eigenvalue of A if and only if the map represented by $A - \lambda I_n$ is not an isomorphism.