Lecture 7

Midterm 1 : 9/28.

- Lecture Notes. - Practice Exam.

Claim Row operations do not affect the column rank of a matrix.

Proof: Consider $A\vec{x} = \vec{0}$. $A = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_m \\ \downarrow & & \downarrow \end{pmatrix}$

$$x_1\vec{v}_1 + \dots + x_m\vec{v}_m = \vec{0}.$$

The solutions $\vec{x}$ give us relations on the columns of A. and at the same time the columns of the RREF of A.

rref(A).

$\Rightarrow$ same linear dependencies.

$\Rightarrow$ same column rank (size of a largest linearly independent set.) $\triangleright$

We can use rref to find dependencies in sets of vectors. Find a ...

Find a basis for

$$\begin{pmatrix}1\\1\\0\end{pmatrix} = \vec{v}_1, \quad \begin{pmatrix}2\\1\\1\end{pmatrix} = \vec{v}_2, \quad \begin{pmatrix}1\\0\\1\end{pmatrix} = \vec{v}_3.$$

Then $A = \begin{pmatrix}1 & 2 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$ rref(A) ?

$$\to \begin{pmatrix}1 & 2 & 1\\ 0 & -1 & -1\\ 0 & 1 & 1\end{pmatrix} \to \begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} \xrightarrow{r_3 + (-1)\cdot r_2} \begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}$$

$$\to \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} = \text{rref}(A).$$

This tells us that $(\vec{v}_1, \vec{v}_2)$ are linearly independent, and that $-\vec{v}_1 + \vec{v}_2 = \vec{v}_3$

$\Rightarrow$ $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\1\\1\end{pmatrix}$ are a basis for the column span of A.

$\Rightarrow$ $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ a basis for the column span of rref(A).

In general the set of solutions to

$A\vec{x} = \vec{w}$ is not a vector space.

When $\vec{w} = \vec{0}$ it is always a vector space.

Suppose that $U, W \subsetneq \mathbb{R}^3$. proper subspaces

Show that and $U \neq W$.

$\dim(U \cap W) \leq 1$.

For any two subspaces, show that $U \cap W$ is a subspace.

If $\vec{x}, \vec{y} \in U \cap W$

need to show that $\vec{x} + \vec{y} \in U \cap W$.

since $\vec{x}, \vec{y} \in U$ we have $\vec{x} + \vec{y} \in U$.

" " " W " " $\vec{x} + \vec{y} \in W$.

$\Rightarrow \vec{x} + \vec{y} \in U \cap W$.

Same for scalar multiples. If $\vec{x} \in U \cap W$, and $r \in \mathbb{R}$

$\ldots\ldots, \quad r\cdot\vec{x} \in U \cap W$

Lecture 8 9/23 Review.

If we want to know if $B = (1+x,\ x-1,\ x^2+x)$ spans $\mathcal{P}_2$. We know $\dim(\mathcal{P}_2) = 3$ so all we need is that B is linearly indep.

Let $E = (1, x, x^2)$ be a (standard) basis for $\mathcal{P}_2$.

Then $B = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_E, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}_E, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_E \right)$

$\hookrightarrow \mathrm{Rep}_E(1+x) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_E$

because $1+x = 1\cdot 1 + 1\cdot x + 0\cdot x^2$.

RREF of $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\Rightarrow B$ is a linearly indep. set.

Let $\mathcal{B} = \left(\begin{pmatrix}1\\1\\0\end{pmatrix}_E, \begin{pmatrix}-1\\1\\0\end{pmatrix}_E, \begin{pmatrix}0\\1\\1\end{pmatrix}_E, \begin{pmatrix}-1\\3\\2\end{pmatrix}_E \right)$

RREF $\begin{pmatrix}1 & -1 & 0 & -1\\ 1 & 1 & 1 & 3\\ 0 & 0 & 1 & 2\end{pmatrix}$ is $\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ \end{array} \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\end{pmatrix}$

linearly independent

$\vec{v}_4 = 0\cdot\vec{v}_1 + 1\vec{v}_2 + 2\cdot\vec{v}_3$

RREF $\begin{pmatrix}-1 & 0 & -1 & 1\\ 1 & 1 & 3 & 1\\ 0 & 1 & 2 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$

We can have an RREF such as

$$\begin{pmatrix} \boxed{1} & 2 & 0 & 0 & 0 & 0\\ 0 & 0 & \boxed{1} & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & \boxed{1} & 3\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ basis of column span looks like

$\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\\vdots\end{pmatrix}, \dots \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$

Question: is $\begin{pmatrix}1\\2\\3\end{pmatrix}$ in the span of $\begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}$?

$\iff$ is there a solution

to

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Find rref of $\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{array}\right)$

$\left(\begin{array}{cc|c} & & \\ 0 & 0 & 1 \end{array}\right)$ ✗

$\left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{array}\right)$ ✓

yes if no leading 1s in last column.

If augmented matrix looks like

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}$$

$x_1 + 2x_4 = 1$

$x_2 + 3x_4 = 2$

$x_3 + 2x_5 = 1$

let $x_4 = s \quad s \in \mathbb{R}$ $\qquad x_5 = t \quad t \in \mathbb{R}$

Then $x_1 = 1 - 2s \qquad x_2 = 2 - 3s$

$x_3 = 1 - 2t.$

Chapter Three. HW 4

Consider a map $\boxed{f: V \to W}$. V, W are vector spaces.

this indicates that f is a function that maps elements of V to elements of W.

We say that f is a homomorphism if it "preserves structure".

$f = \cos x \quad f: \mathbb{R} \to \mathbb{R}$.

or $f: \mathbb{R} \to [0,1]$.

We say that $f: V \to W$ is a homomorphism if

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \forall \vec{v}_1, \vec{v}_2 \in V$$

$$\text{and} \quad f(r \cdot \vec{v}_1) = r \cdot f(\vec{v}_1) \quad \forall r \in \mathbb{R}, \ \vec{v}_1 \in V.$$

Lemma 1.11: This is equivalent to saying that

$$f(\underbrace{r_1\vec{v}_1 + r_2\vec{v}_2}_{\in V}) = r_1 f(\vec{v}_1) + r_2 f(\vec{v}_2)$$

$\in W$

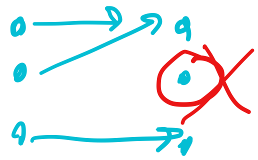
$$\forall r_1, r_2 \in \mathbb{R}, \ \vec{v}_1, \vec{v}_2 \in V.$$

A function $f: V \to W$ is an isomorphism if it is a homomorphism and is injective and surjective, i.e. bijective.

each element of

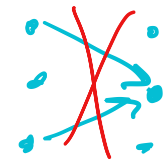
each element of W is mapped

W is mapped to at least once.



to at most once.

$f(\vec{v_1}) = f(\vec{v_2}) \iff \vec{v_1} = \vec{v_2}$.



An isomorphism $f: V \to V$ is called an automorphism.

<u>Example</u>: Scaling of $\mathbb{R}^2$. $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(\vec{v}) = 2\vec{v}$. E.g. $f\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = \begin{pmatrix}2\\4\end{pmatrix}$

$\begin{pmatrix}1\\2\end{pmatrix} \mapsto \begin{pmatrix}2\\4\end{pmatrix}$

Check that this is a homomorphism:

$$\begin{aligned} f(r_1\vec{v_1} + r_2\vec{v_2}) &= 2(r_1\vec{v_1} + r_2\vec{v_2}) = 2r_1\vec{v_1} + 2r_2\vec{v_2} \\ &= r_1(2\vec{v_1}) + r_2(2\vec{v_2}) \\ &= r_1 f(\vec{v_1}) + r_2 f(\vec{v_2}) \end{aligned}$$

$\forall r_1, r_2 \in \mathbb{R}$
$\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$.

Let's check that this is an automorphism/isomorphism

- injective: Suppose that $f(\vec{v_1}) = f(\vec{v_2})$

$\Rightarrow \quad 2\vec{v_1} = 2\vec{v_2}$

$\Rightarrow$

$\Rightarrow \quad \vec{v}_1 = \vec{v}_2 \quad \forall \vec{v}_1, \vec{v}_2 \in \mathbb{R}^2.$

- surjective: For any $\vec{w} \in V$ we have that $\vec{w} = 2\left(\frac{1}{2}\vec{w}\right) = f\left(\underbrace{\tfrac{1}{2}\vec{w}}_{\in V}\right)$.

This is a linear transformation:

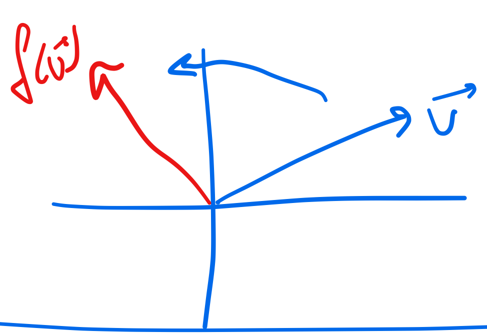
$$f(\vec{v}) = A\vec{v} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\vec{v}. = 2\vec{v}.$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Example: rotation. $f : \mathbb{R}^2 \to \mathbb{R}^2$

f rotates $\vec{v}$ over $90°$ counterclockwise.

$$f(\vec{v}) = A\vec{v} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$$

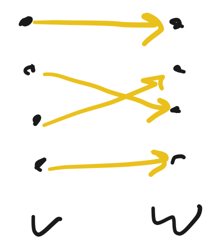


Section I.2 (of Chapter Three).

Claim The inverse of an isomorphism is

also an isomorphism.

Pf: Suppose $f: V \to W$ is an isomorphism. Since f is a bijection, there is an inverse f^{-1} that is also a bijection.



We have $f^{-1}: W \to V$.

Let $f(\vec{v_1}) = \vec{w_1}$ $f(\vec{v_2}) = \vec{w_2}$

$\Rightarrow f^{-1}(\vec{w_1}) = \vec{v_1}$ $f^{-1}(\vec{w_2}) = \vec{v_2}$.

$$f^{-1}(r_1\vec{w_1} + r_2\vec{w_2}) =$$

$$= f^{-1}(r_1 f(\vec{v_1}) + r_2 f(\vec{v_2})) = f^{-1}(f(r_1\vec{v_1} + r_2\vec{v_2})) =$$

$$r_1\vec{v_1} + r_2\vec{v_2} = r_1 f^{-1}(\vec{w_1}) + r_2 f^{-1}(\vec{w_2})$$

$\forall \vec{w_1}, \vec{w_2} \in W$. $\square$

Equivalence Relations.

A relation on a set A is a set of ordered pairs from A.

We call R an equivalence relation, denoted $a \sim b$ if

Example:

$A = \{1, 2, 3, 4\}$

$R = \{(1,2), (3,3),$

$(a,b) \in R$, $\quad (1,4), (4,1) \}$

if R is
- reflexive: $a \sim a \quad \forall a \in A$.
- symmetric: $a \sim b \iff b \sim a \quad \forall a,b \in A$.
- transitive: $a \sim b,\ b \sim c \implies a \sim c \quad \forall a,b,c \in A$.

Theorem 2.2 (page 184)

Isomorphism defines an equivalence relation on vector spaces.

$$\overbrace{\vec{u}_1 \ldots \vec{u}_p \ \vec{w}_1 \ldots \vec{w}_t}^{p+t \le n}$$

$\dim V = n$

$\dim U = p$

$\dim W = q$

$$\underbrace{\vec{w}^*_{t+1} \ldots \vec{w}^*_q}_{\in U}$$

$p+t \le n$.

$$\vec{w}_{t+1} \in [(\underbrace{\vec{u}_1 \ldots \vec{u}_p}, \underbrace{\vec{w}_1 \ldots \vec{w}_t})]$$

$$\vec{w}_{t+1} = \vec{u} + \vec{w}$$

$b_1 \vec{w}_1 + \ldots + b_t \vec{w}_t$

$$\vec u = \vec w_{t+1} - \vec w$$

$$= \underbrace{a_1\vec u_1 + \dots + a_p\vec u_p + b \dots}_{\overset{*}{w_{t+1}}}$$

$$\underbrace{\vec u_1 \dots \vec u_p \quad \vec w_1 \dots \vec w_t}_{\le n} \quad \vec w_{t+1} \dots \vec w_q$$

$$\vec w_{t+2}$$

$p + t \le n$

$t \le n - p$

$$\dim U \cap W = q - t \ge \underset{\dim W}{q} - \underset{\dim V}{n} + \underset{\dim U}{p}$$

$$\overbrace{\vec u_1 \dots \vec u_p}^{\text{basis for } U} \quad \overbrace{\vec w_1 \dots \vec w_t \quad \underbrace{\overset{*}{\vec w_{t+1}} \dots \overset{*}{\vec w_q}}_{\text{linearly ind. in } U \cap W}}^{\text{basis for } W \ (\dim W)}$$

~~basis for~~ $U \cap W$.

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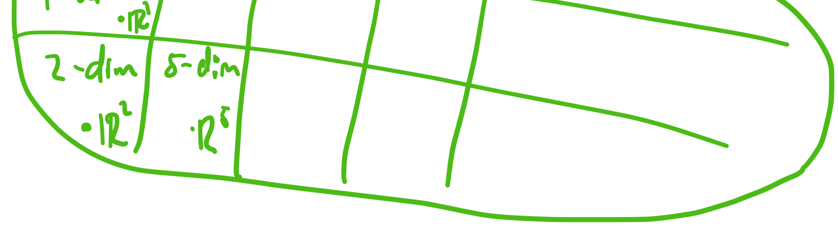
Thm 2.2 (p. 184) Isomorphism defines an equivalence relation on vector spaces.

Notation: $V \cong W \iff V$ is isomorphic to W.

gives a classification:



Theorem 2.3: $V \cong W \iff \dim V = \dim W$.

(lemma 2.4) "$\Rightarrow$" Suppose $f: V \to W$ is an isomorphism. <u>Idea</u>: Let B_V be a basis for V, put it through f and obtain a basis for W.

$B_V = (\beta_1 \ldots \beta_n)$ basis for V.

Consider the set $(f(\beta_1), f(\beta_2) \ldots f(\beta_n)) = B_W$.

For any $\vec{w} \in W$ we have a unique $\vec{v} \in V$ such that $f(\vec{v}) = \vec{w}$. $(\vec{v} = f^{-1}(\vec{w}).)$

$$\vec{v} = v_1 \vec{\beta}_1 + v_2 \vec{\beta}_2 + \ldots + v_n \vec{\beta}_n = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}_{B_V}.$$

$$\underset{\vec{w}}{\underset{\shortparallel}{f(\vec{v})}} = f(v_1 \vec{\beta}_1 + \ldots + v_n \vec{\beta}_n) = v_1 f(\vec{\beta}_1) + \ldots + v_n f(\vec{\beta}_n).$$

$$\Rightarrow W = [B_W].$$

Recall that $f(\vec{0}) = \vec{0}$ for any homomorphism.

$\Rightarrow$ relations on vectors are also preserved:

$$a_1 \vec{x_1} + \dots + a_m \vec{x_m} = \vec{0}$$

$$f(a_1\vec{x_1} + \dots + a_m \vec{x_m}) = \vec{0}$$

$$a_1 f(\vec{x_1}) + \dots + a_m f(\vec{x_m}) = \vec{0}.$$

In W this implies that any relation on B_W is also a relation on B_V and vice versa.

B_V linearly independent $\iff$ B_W is linearly independent.
(True)

$\Rightarrow$ B_W is a basis of W.

$$|B_V| = |B_W|.$$

$$\dim V = \dim W.$$

(Lemma 2.5) "$\Leftarrow$" Suppose $\dim V = \dim W = n$.

Then want to show that $V \cong W$.

Idea: let $B_V = (\vec{\alpha_1}, \dots \vec{\alpha_n})$ basis for V
$B_W = (\vec{\beta_1}, \dots, \vec{\beta_n})$ basis for W.

Natural isomorphism to try: $f: V \to W$

let $f(\vec{v}) = f\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{B_V}\right) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{B_W}$.

Then show that this is indeed an isomorphism. ∎

Recall that an automorphism is a isomorphism
$f: V \to V$.

homomorphisms : linear maps
homomorphisms : linear transformations.
$V \to V$.

Linear maps can always be written as matrix functions: $f(\vec{v}) = A\vec{v}$

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$.

$$f(\vec{v}) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \vec{v} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_1 + v_2 \end{pmatrix}.$$

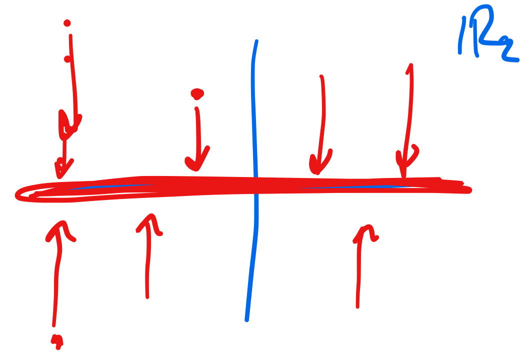
$\vec{v}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and let $\vec{v}_{i+1} = f(\vec{v}_i)$.

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 8 \\ 13 \end{pmatrix}, \ldots$ gives us the Fibbonacci numbers.

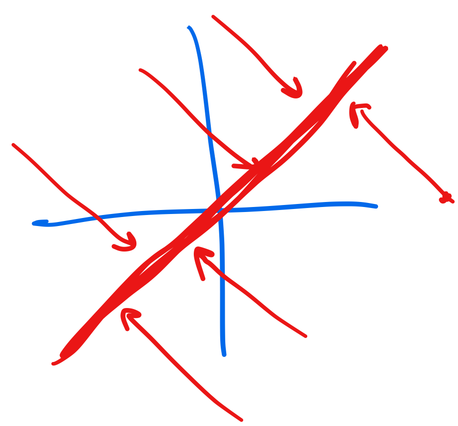
Example: Projections: $f: \mathbb{R}^2 \to \mathbb{R}^2$

Let $f(\vec{v}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{v} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix}$.

Clearly not invertible.



Could also have a projection to $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$:



Example $\mathcal{P}_3 = \{ a + bx + cx^2 + dx^3 \mid a, b, c, d \in \mathbb{R} \}$,

Then $\frac{d}{dx}$ is a linear transformation:

$$\frac{d}{dx}\left(a_1 \cdot p_1(x) + a_2 \cdot p_2(x) \right) = a_1 \cdot \frac{dp_1(x)}{dx} + a_2 \frac{dp_2(x)}{dx}.$$

$B = (1, x, x^2, x^3)$

$a + bx + cx^2 + dx^3 \longmapsto b + 2c\,x + 3d\,x^2$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_B \longmapsto \begin{pmatrix} b \\ 2c \\ 3d \\ 0 \end{pmatrix}_B$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 3d \\ 0 \end{pmatrix}$$

We can also express this as a map $\mathcal{P}_3 \to \mathcal{P}_2$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 3d \end{pmatrix}.$$

Example: $S = \{ a \sin x + b \cos x \mid a, b \in \mathbb{R} \}$

Consider the map $\frac{d}{dx}$. Is it linear?

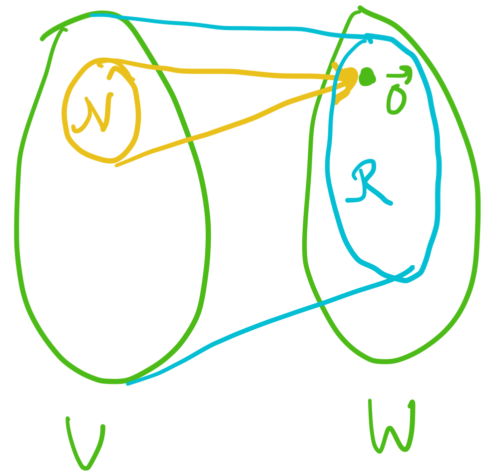
Or consider: $f : S \to \mathbb{R}^2 \quad a \sin x + b \cos x \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$.

Definition: The range space of a homomorphism

$f: V \to W$ is the set

$$\mathcal{R}(f) = \{ \vec{w} \mid \exists\, \vec{v} \in V \text{ such that } f(\vec{v}) = \vec{w} \} =$$

$$\{ f(\vec{v}) \mid \vec{v} \in V \}.$$



The <u>null space</u> is the set

$$\mathcal{N}(f) = \{ \vec{v} \in V \mid f(\vec{v}) = \vec{0} \}.$$

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Let $f: V \to W$ be a linear map.

<u>Claim</u> the range space $\mathcal{R}(f)$ is a subspace of W.

<u>Proof</u>: Let $\vec{w}_1, \vec{w}_2 \in \mathcal{R}(f)$ and $c_1, c_2 \in \mathbb{R}$

Let $\vec{v}_1, \vec{v}_2 \in V$ be vectors such that

$f(\vec{v}_1) = \vec{w}_1$ and $f(\vec{v}_2) = \vec{w}_2$ then

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) = c_1\vec{w}_1 + c_2\vec{w}_2.$$

$$\Rightarrow \quad c_1\vec{w}_1 + c_2\vec{w}_2 \in \mathcal{R}(f). \quad \square$$

Claim: $N(f)$ is a subspace of V.

Proof let $\vec{v}_1, \vec{v}_2 \in N(f)$ and $c_1, c_2 \in \mathbb{R}$, then we have $f(\vec{v}_1) = \vec{0}$ $f(\vec{v}_2) = \vec{0}$, and therefore

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) = \vec{0}.$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 \in N(f). \quad \square$$

Thm 2.14 p205 $\underbrace{\dim V}_{=n} = \overbrace{\underbrace{\dim \mathcal{R}(f)}_{=?}}^{\text{rank}} + \overbrace{\underbrace{\dim N(f)}_{=k}}^{\text{nullity}}.$

Pf let $\langle \vec{\beta}_1 \dots \vec{\beta}_k \rangle$ be a basis for $N(f)$.

We can extend this to a basis

$$\langle \vec{\beta}_1, \dots, \vec{\beta}_k, \vec{\beta}_{k+1}, \dots \vec{\beta}_n \rangle \text{ of } V.$$

Consider the set $\langle f(\vec{\beta}_{k+1}), \dots, f(\vec{\beta}_n) \rangle$ in W.

We will show this is a basis for $\mathcal{R}(f)$.

Let $\vec{w} \in \mathcal{R}(f)$ then $\vec{w} = f(\vec{v})$ for some $\vec{v} \in V$.

and $\vec{v} = a_1\vec{\beta}_1 + a_2\vec{\beta}_2 + \dots + a_n\vec{\beta}_n$.

$$\Rightarrow \vec{w} = \underbrace{a_1 f(\vec{\beta}_1) + \ldots + a_k f(\vec{\beta}_k)}_{=\vec{0}} + a_{k+1} f(\vec{\beta}_{k+1}) \ldots + a_n f(\vec{\beta}_n).$$

$$\vec{w} = a_{k+1} f(\vec{\beta}_{k+1}) + \ldots + a_n f(\vec{\beta}_n).$$

$$\Rightarrow [\langle f(\vec{\beta}_{k+1}), \ldots f(\vec{\beta}_n) \rangle] = \mathcal{R}(f).$$

Still need linear independence.

Suppose that we have a nontrivial relation in $\mathcal{R}(f) \subseteq W$

$$c_{k+1} f(\vec{\beta}_{k+1}) + \ldots + c_n f(\vec{\beta}_n) = \vec{0}$$

This implies that

$$f(c_{k+1} \vec{\beta}_{k+1} + \ldots + c_n \vec{\beta}_n) = \vec{0}$$

$$\Rightarrow c_{k+1} \vec{\beta}_{k+1} + \ldots + c_n \vec{\beta}_n \in \mathcal{N}(f)$$

$$\shortparallel$$

$$c_1 \vec{\beta}_1 + c_2 \vec{\beta}_2 + \ldots + c_k \vec{\beta}_k .$$

Contradiction, since $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for V.

$$\Rightarrow \langle f(\vec{\beta}_{k+1}), \ldots, f(\vec{\beta}_n) \rangle \text{ is a basis for } \mathcal{R}(f).$$

$\Rightarrow \dim \mathcal{R}(f) = n-k.$ $\square$

Example $f: \mathcal{P}_3 \to \mathcal{P}_2$ $\quad f(p(x)) = \dfrac{dp(x)}{dx}$

$$f(a+bx+cx^2+dx^3) = b+2cx+3dx^2. \quad a,b,c,d \in \mathbb{R}.$$

What is the null space $\mathcal{N}(f)$?

Polynomials with 0 derivative $p(x) = a$.

Basis for $\mathcal{N}(f)$: $\langle 1 \rangle$.

What is $\mathcal{R}(f)$?

We see that every $p(x) \in \mathcal{P}_2$ is a derivative of some $q(x) \in \mathcal{P}_3$.

We have $\langle 1, x, x^2 \rangle$ as a basis for $\mathcal{R}(f)$.

$$= \langle f(x), f(\tfrac{1}{2}x^2), f(\tfrac{1}{3}x^3) \rangle.$$

Let $f: V \to W$ be a linear map.

Let $B = \langle \vec{\beta}_1 \ldots \vec{\beta}_n \rangle$ be a basis for V.

Then $f(\vec{\beta}_1), \dots, f(\vec{\beta}_n)$ completely determines f.

Since for any $\vec{v} \in V$ we have

$$f(\vec{v}) = f(a_1\vec{\beta}_1 + \dots + a_n\vec{\beta}_n) = a_1 f(\vec{\beta}_1) + \dots + a_n f(\vec{\beta}_n).$$

Let $f : \mathcal{M}_{2\times 2} \to \mathcal{P}_3$ such that (p. 200)

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + (2c-d)x + bx^2 + ax^3.$$

What is $\mathcal{N}(f)$?

$$\mathcal{N}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a = b = 0, & 2c-d=0 \\ & 2c=d \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 \\ c & 2c \end{pmatrix} = c \cdot \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}.$$

Basis $\left\langle \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \right\rangle$.

What is $\mathcal{R}(f)$?

$c' = 2c-d$

$$\mathcal{R}(f) = \left\{ a + (2c-d)x + bx^2 + ax^3 \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

$= \{ a(1+x^3) + b x^2 + c' x \mid a, b, c' \in \mathbb{R} \}$

Basis : $\langle 1+x^3, x^2, x \rangle$.

Lecture 13 10/12.

Retake M1 is due 10/19.

M2: 10/26.

Thm 2.20 (p.207)

Linear map $f: V \to W$, $\dim V = n$.

The following are equivalent:

(1) f is one-to-one → "1-1" or injective

(2) f has an inverse function f^{-1} which is a linear map $W \to V$.

(3) nullity $(f) = 0$

(4) rank $(f) = n$

(5) If $\langle \beta_1 \ldots \beta_n \rangle$ is a basis for V then $\langle f(\beta_1), \ldots, f(\beta_n) \rangle$ is a basis for $\mathcal{R}(f)$.

Even if a map f is not invertible, we

can always define the "inverse image":

$$f^{-1}(S) = \{ \vec{v} \in V \mid f(\vec{v}) \in S \}.$$

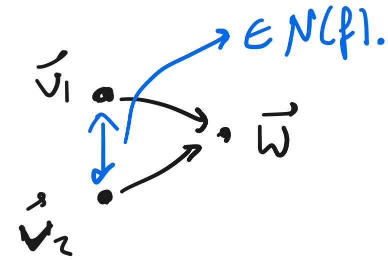
$S \subseteq W$ $\qquad f^{-1}(\vec{w}) = \{ \vec{v} \in V \mid f(\vec{v}) = \vec{w} \}.$

For example $f^{-1}(\mathcal{R}(f)) = V.$

$$f^{-1}(\vec{0}) = \mathcal{N}(f).$$

Linear map $f: V \to W$. If $\text{nullity}(f) > 0$.

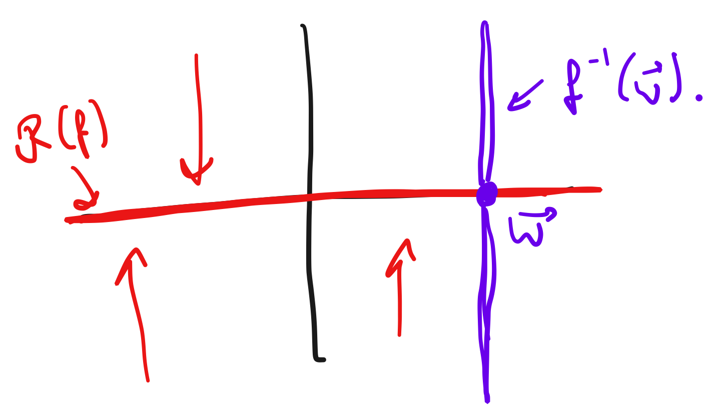
and $f(\vec{v}_1) = f(\vec{v}_2)$ for $\vec{v}_1 \neq \vec{v}_2$



Then $\vec{v}_1 - \vec{v}_2 \in \mathcal{N}(f)$, because

$$f(\vec{v}_1 - \vec{v}_2) = f(1 \cdot \vec{v} + (-1) \cdot \vec{v}_2) = f(\vec{v}_1) - f(\vec{v}_2) = \vec{w} - \vec{w} = \vec{0}.$$

Example: Projection



$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

Then $\mathcal{R}(f) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

$\mathcal{N}(f) = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].$

Example: $f(p(x)) = d/dx\, p(x)$ $\quad f: \mathcal{P}_2 \to \mathcal{P}_2.$

What is $f^{-1}(1+2x)$?

This is all $p(x) = a + bx + cx^2$ such that $\frac{d}{dx} p(x) = 1 + 2x$. These look like:

$$q(x) = C + x + x^2 \quad \text{for any } C \in \mathbb{R}.$$

Indeed $\mathcal{N}(f) = [1] \to$ Namely, constant functions $p(x) = c$.

Linear map $f: V \to W$.

Let $B = \langle \vec{\beta}_1 \dots \vec{\beta}_n \rangle$ be a basis for V.

Then $f(\vec{\beta}_1), f(\vec{\beta}_2) \dots f(\vec{\beta}_n)$ defines f,

since $f(\vec{v}) = f(c_1\vec{\beta}_1 + \dots + c_n\vec{\beta}_n)$
$\vec{v} \in V$
$= c_1 f(\vec{\beta}_1) + \dots + c_n f(\vec{\beta}_n)$.

Suppose D is a basis for W, and $\dim W = m$.

Then every $f(\vec{v}) = \vec{w}$ has a representati

... $f(\vec{v}) = \vec{w}$ has a representation in terms of D.

We can let $\mathrm{Rep}_D(f(\vec{\beta}_1)) = \begin{pmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{m,1} \end{pmatrix}$

$\mathrm{Rep}_D(f(\vec{\beta}_2)) = \begin{pmatrix} h_{1,2} \\ h_{2,2} \\ \vdots \\ h_{m,2} \end{pmatrix} \cdots \mathrm{Rep}_D(f(\vec{\beta}_n)) = \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ \vdots \\ h_{m,n} \end{pmatrix}.$

Then

$$f(\vec{v}) = \begin{pmatrix} h_{1,1} & \cdots & h_{1,n} \\ h_{2,1} & & \vdots \\ \vdots & & \vdots \\ h_{m,1} & \cdots & h_{m,n} \end{pmatrix} \mathrm{Rep}_B(\vec{v})$$

gives us $\mathrm{Rep}_D(\vec{w})$, since:

$$\begin{pmatrix} h_{1,1} & \cdots & h_{1,n} \\ \vdots & & \vdots \\ h_{m,1} & \cdots & h_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1 \begin{pmatrix} h_{1,1} \\ \vdots \\ h_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} h_{1,n} \\ \vdots \\ h_{m,n} \end{pmatrix}$$

$$= c_1 \cdot \left(f(\vec{\beta}_1)\right)_D + \cdots + c_n \left(f(\vec{\beta}_n)\right)_D .$$

Example ...

Example $f: P_2 \to P_2$ $\quad f(p(x)) = \frac{d}{dx} p(x)$

$B = \langle 1, x, x^2 \rangle = D$ $\quad \vec{\beta}_1 = 1 \quad \vec{\beta}_2 = x \quad \vec{\beta}_3 = x^2$.

$$f(\vec{\beta}_1) = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_D \quad f(\vec{\beta}_2) = 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_D \quad f(\vec{\beta}_3) = 2x = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}_D.$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$p(x) = 2 + 3x - 4x^2 = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}_B$$

$$\frac{d}{dx} p(x) = 3 - 8x = \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix}_D$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}_B = \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix}_D.$$

Instead, now let $B = \langle 1, 1+x, x^2 \rangle$

$D = \langle 1, 1-x \rangle$.

$f(p(x)) = \frac{d}{dx} p(x)$

$\vec{\beta}_1 = 1 \quad \vec{\beta}_2 = 1+x \quad \vec{\beta}_3 = x^2$

$f: P_2 \to P_1$.

$$f(\vec{\beta}_1) = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_D \quad f(\vec{\beta}_2) = 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_D$$

$$f(\vec{\beta}_3) = 2x = 2 \cdot 1 + (-2) \cdot (1-x)$$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix}_D.$$

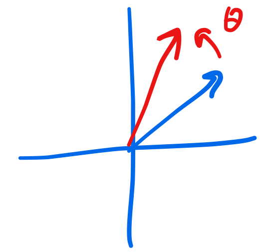
$$f(p(x)): \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{Rep}_D(p(x)) = \ldots$$

$\mathrm{Rep}_{B,D}(f(p(x))) = \underbrace{}_{}\ \mathrm{Rep}_{B,D}\ \mathrm{Rep}_B(p(x)) = \mathrm{Rep}_D(f(p(x)))$.

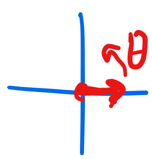
Rotation of $\mathbb{R}^2$. $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

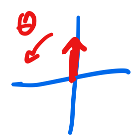
Standard bases. via

$\vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$.



$f(\vec{e}_1) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ $f(\vec{e}_2) = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$.

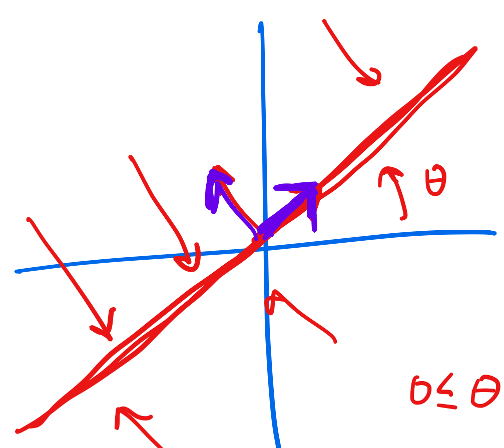




$$\mathrm{Rep}_{E,E}(f) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}_{E,E}.$$

Projections $f_\ell : \mathbb{R}^2 \to \mathbb{R}^2$

by projecting to ℓ



$0 \le \theta < \pi$.

Use basis

$B = \langle \vec{\beta}_1 = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \vec{\beta}_2 = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \rangle$ $\mathcal{N}(f) = [\vec{\beta}_2]$

$f(\vec{\beta}_1) = \vec{\beta}\ \ \ (1)$ $\mathcal{R}(f) = [\vec{\beta}]$

$f(\vec{\beta}_1) = \vec{\beta}_1 = \binom{1}{0}_B$ $\qquad$ $(f) = [\vec{\beta}_1]$.

$f(\vec{\beta}_2) = 0 = \binom{0}{0}_B$.

$\mathrm{Rep}_{B,B}(f) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{B,B}$.

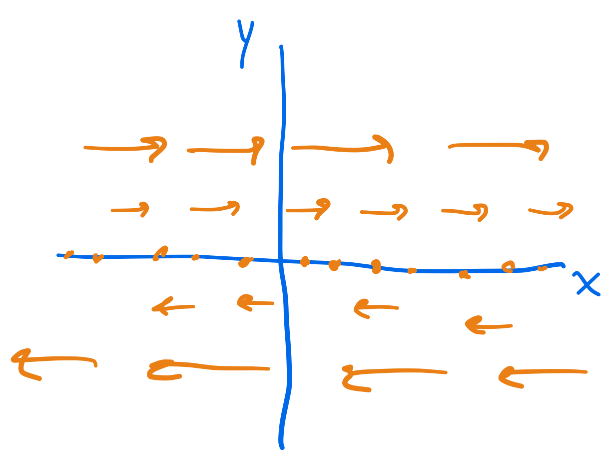
10/14

M1 retake due 10/19

M2 on 10/26.

Example: A shear is a transformation of $\mathbb{R}^2$ where we add a multiple of one coefficient to another. $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Every linear map can be represented as

a matrix (depends on choice of bases for domain / codomain).

Conversely, every matrix represents a linear map.

As we have seen, the span of the columns of a matrix A is the set of all vectors of the form $A\vec{v}$ $\Rightarrow$ the column span is the range space (expressed in some basis D).

$$\Rightarrow \quad \text{rank } A = \text{rank } f \quad \text{if } A \text{ represents } f.$$

Combining Matrices / Linear maps.

If $f, h : \mathbb{R}^n \to \mathbb{R}^m$ then let $f+h : \mathbb{R}^n \to \mathbb{R}^m$ be defined as $(f+h)(\vec{v}) = f(\vec{v}) + h(\vec{v})$.

If $f(\vec{v}) = A\vec{v}$ $\quad h(\vec{v}) = B\vec{v}$

when we omit the basis assume standard basis.

Then $(f+h)(\vec{v}) = (A+B)\vec{v} = A\vec{v} + B\vec{v}$.

where

$$A+B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

$= ($

$$= \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}.$$

Function composition $f: \mathbb{R}^n \to \mathbb{R}^p$ $h: \mathbb{R}^p \to \mathbb{R}^m$.

Let $h \circ f$ be the composition of functions:

$$h \circ f(\vec{v}) = h(f(\vec{v})).$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "h \circ f"'] & \mathbb{R}^p \arrow[r, "h"] & \mathbb{R}^m \end{tikzcd}$$

$f(\vec{v}) = A\vec{v}$ $\qquad$ $h(\vec{w}) = B\vec{w}$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1,n} \\ \vdots & & \\ a_{p,1} & \cdots & a_{pn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mp} \end{pmatrix}$$

$$h \circ f(\vec{v}) = (BA)\vec{v}.$$

The ij^{th} entry of BA is the dot product of the i^{th} row of B with the j^{th} column of A.

Example

$$\begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -1 & -3 \\ 5 & 3 \end{pmatrix}$$

$= 1\cdot 2 + (-1)\cdot 1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$5 = 2\cdot 2 + 1\cdot 1$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\underset{m \times p}{B}\;\underset{p \times n}{A} = \underset{m \times n}{C}$$

Recall the dot product of two vectors $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots v_n w_n. \in \mathbb{R}.$$

$f : \mathbb{R}^3 \to \mathbb{R}^2$ $\qquad h : \mathbb{R}^2 \to \mathbb{R}^3$

$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ $\qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{pmatrix}.$

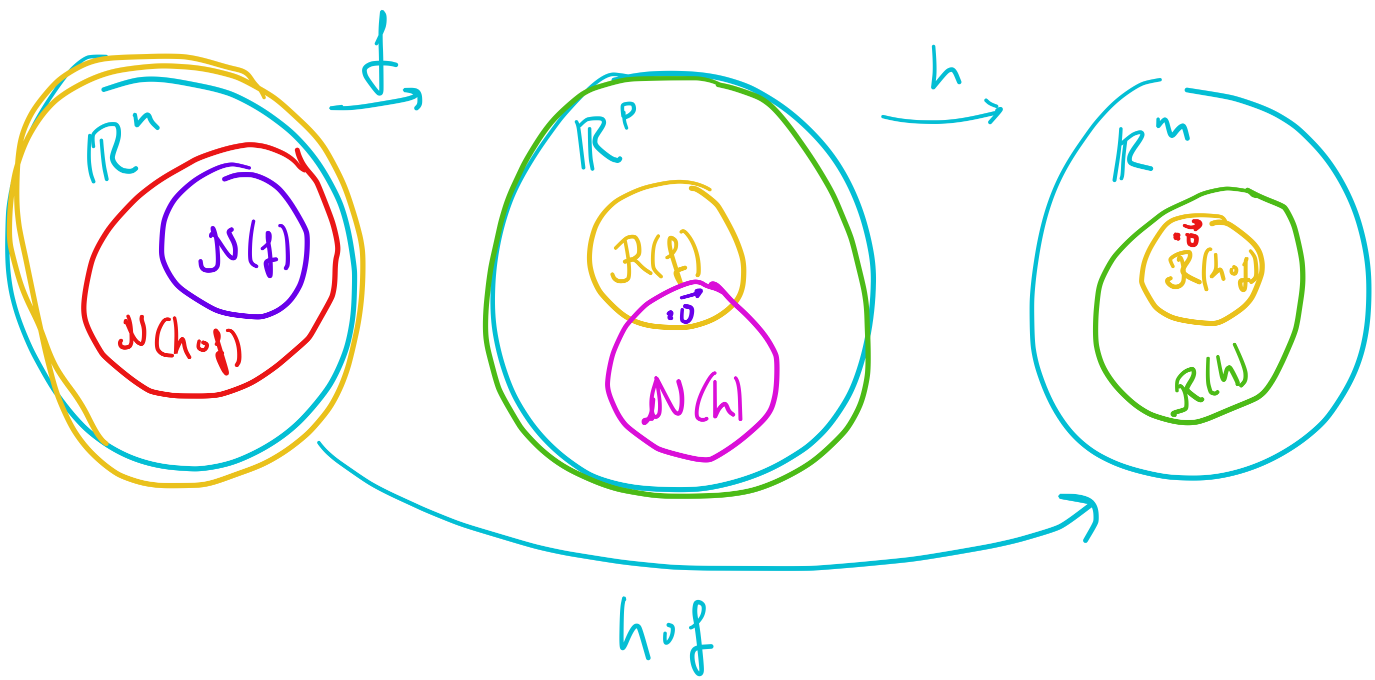
$$\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{h} \mathbb{R}^3$$

$$h \circ f(\vec{v}) = (BA)\vec{v} = B(A\vec{v})$$

$$BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 4 & 4 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

then $$(BA)\vec{v} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 23 \end{pmatrix}$$

$$B(A\vec{v}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 23 \end{pmatrix}$$



Suppose that $\vec{z} \in \mathbb{R}^m$ and $\vec{z} \in \mathcal{R}(h\circ f)$. Then there is some $\vec{v} \in \mathbb{R}^n$ s.t. $(h\circ f)(\vec{v}) = h(f(\vec{v})) = \vec{z}$. Let $\vec{w} = f(\vec{v}) \in \mathbb{R}^p$ then $h(\vec{w}) = \vec{z}$. $\Rightarrow \vec{z} \in \mathcal{R}(h)$.

$$\mathcal{R}(h\circ f) \subseteq \mathcal{R}(h).$$

If $\vec{v} \in \mathcal{N}(f)$ then $f(\vec{v}) = \vec{0}$.

$$\Rightarrow (h\circ f)\vec{v} = h(f(\vec{v})) = h(\vec{0}) = \vec{0}.$$

$$\Rightarrow \quad \vec{v} \in \mathcal{N}(h\circ f)$$

$$\Rightarrow \quad \mathcal{N}(f) \subseteq \mathcal{N}(h\circ f).$$

Oct 19

Midterm 2 Oct 26.
(no HW due)
Chapter Two and Three
(sections I–III).

Proof by contradiction

Suppose we want to show $\underbrace{A \Rightarrow B}$ by contradiction.

A and not B is impossible.

Proof. Suppose A, and suppose (for the sake of that not B. contradiction)

Then this contradicts something

- For example $\Rightarrow$ not A.

- or $\Rightarrow$ $0 = 1$.

- if we conclude B that does contradict not B, but we probably could have written a direct proof $A \Rightarrow B$.

Inverses (IV.4 p. 254)

$f : \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{f} \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ $\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{h} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ $\quad$ $h : \mathbb{R}^2 \mapsto \mathbb{R}^3$.

$h \circ f : \mathbb{R}^3 \to \mathbb{R}^3$ $\quad$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{h \circ f} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ $\quad$ not surjective

$f \circ h : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad$ $\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{f \circ h} \begin{pmatrix} x \\ y \end{pmatrix}$. $\quad$ identity.

We say that f is a left-inverse of h.

h is a right-inverse of f

g is a right inverse of f.

The matrix that gives the identity map $\mathbb{R}^n \to \mathbb{R}^n$ is called the identity matrix I_n.

$$I_n = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

h has matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

f has matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

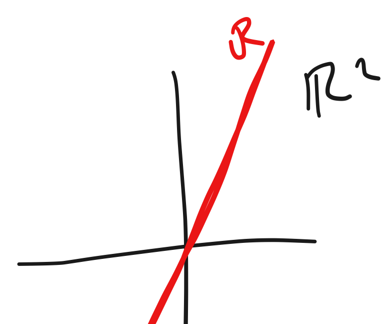
$f \circ h$ has matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$h \circ f$ has matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3$.

f: $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Does $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ have an inverse?



We have $\text{rank}(f) + \text{null}(f) = 2$

$= 1$

$\Rightarrow 1$

$\Rightarrow$ not 1-1.

cannot have an inverse.

Does $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = A$ have an inverse?

$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x+y \\ x \end{pmatrix}$

Full-rank, yes.

$\downarrow$

$n \times n$ matrix

and rank $= n$.

We need that

$A^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a-b \end{pmatrix}$

$A^{-1}\begin{pmatrix} x+y \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$AA^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

can $A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $= I_2$.

In general $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

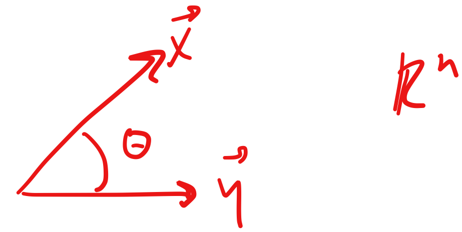
$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Lecture 10/28

Projections The dot product of 2 vectors

... 2 vectors, has something to do with the angle between them:

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos \theta.$$



$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} = \sqrt{\vec{x} \cdot \vec{x}}.$$

$$\cos \theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \cdot \|\vec{y}\|}.$$

Example: Application: correlation coefficient.

Mathematicians	coffee	papers
A	7	8
B	14	10
C	11	8
D	4	6
	$\mu = 9$	$\mu = 8$

Normalize by subtracting the mean:

coffee: $\vec{x} = \begin{pmatrix} -2 \\ 5 \\ 2 \\ -5 \end{pmatrix}$ papers: $\vec{y} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \end{pmatrix}$

Correlation coefficient $= r = \cos \theta = \dfrac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}$

$$= \frac{-2\cdot 0 + 5\cdot 2 + \dots + (-5\cdot -2)}{\sqrt{(-2)^2+5^2+\dots}\cdot\sqrt{0^2+2^2+\dots}} \sim .93.$$

Two vectors $\vec{x}, \vec{y}$ are <u>orthogonal</u> if $\vec{x}\cdot\vec{y}=0$.

Let V be a vector space and U a subspace of V.

$V = \mathbb{R}^n$.

Then we define $U^\perp$ as the <u>orthogonal complement</u> of U: the set (subspace) of vectors in V that are orthogonal to every vector in U.

<u>Facts</u>:

- U and $U^\perp$ have only $\vec{0}$ in common.
- $\dim U + \dim U^\perp = n$
- $(U^\perp)^\perp = U$.

Follows from rank nullity and a projection onto U.

- Every vector $\vec{v} \in \mathbb{R}^n$ can be expressed as $\vec{v} = \vec{v}^{\parallel} + \vec{v}^{\perp}$ such that $\vec{v}^{\parallel} \in U$, $\vec{v}^\perp \in U^\perp$.

$= \mathrm{proj}_U \vec{v}$

An <u>orthonormal basis</u> for $\mathbb{R}^n$ is a basis

$\langle \hat{u}_1, \ldots, \hat{u}_n \rangle$ such that $\hat{u}_i \cdot \hat{u}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$

Standard bases are orthonormal.

Suppose that $\langle \hat{u}_1, \ldots, \hat{u}_p \rangle$ is an orthonorm. basis for U.

$$\text{proj}_U \vec{v} \in U$$

$$\Rightarrow \text{proj}_U \vec{v} = a_1 \cdot \hat{u}_1 + a_2 \hat{u}_2 + \ldots + a_p \hat{u}_p.$$

$$\vec{v} - \text{proj}_U \vec{v} \in U^\perp.$$

$$(\vec{v} - a_1 \hat{u}_1 - a_2 \hat{u}_2 - \ldots - a_p \hat{u}_p) \cdot \hat{u}_1 = 0$$

$$\Rightarrow (\vec{v} - a_1 \hat{u}_1) \cdot \hat{u}_1 = 0$$

since $a_k \hat{u}_k \cdot \hat{u}_1 = 0$ for $k \neq 1$.

$$\Rightarrow \vec{v} \cdot \hat{u}_1 = a_1 \underbrace{(\hat{u}_1 \cdot \hat{u}_1)}_{=1} \qquad a_1 = \vec{v} \cdot \hat{u}_1.$$

Same for $a_2 = \vec{v} \cdot \hat{u}_2 \quad \ldots \quad a_p = \vec{v} \cdot \hat{u}_p$.

Now we have a formula for projections:

$$\text{proj}_U \vec{v} = (\vec{v} \cdot \hat{u}_1) \hat{u}_1 + (\vec{v} \cdot \hat{u}_2) \hat{u}_2 + \ldots + (\vec{v} \cdot \hat{u}_p) \hat{u}_p.$$

We can express this as a matrix as follows.

$$\underbrace{\begin{pmatrix} \uparrow & & \uparrow \\ \hat{u}_1 & \cdots & \hat{u}_p \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \leftarrow & \hat{u}_1 & \rightarrow \\ \leftarrow & \hat{u}_2 & \rightarrow \\ & \vdots & \\ \leftarrow & \hat{u}_p & \rightarrow \end{pmatrix} \vec{v}}$$

$$= (\hat{u}_1 \cdot \vec{v})\hat{u}_1 + \cdots + (\hat{u}_p \cdot \vec{v})\hat{u}_p.$$

$$= \begin{pmatrix} \hat{u}_1 \cdot \vec{v} \\ \hat{u}_2 \cdot \vec{v} \\ \vdots \\ \hat{u}_p \cdot \vec{v} \end{pmatrix}$$

A transpose A^T of a matrix A is reversing the role of rows and columns.

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 4 \end{pmatrix}.$$

Now we find the projection matrix A of a projection onto U by letting

$$Q = \begin{pmatrix} \uparrow & & \uparrow \\ \hat{u}_1 & \cdots & \hat{u}_p \end{pmatrix} \qquad \langle \hat{u}_1, \ldots, \hat{u}_p \rangle \text{ an orthonormal}$$

basis of U

and $A = QQ^T$.