# Lecture 1 Aug 31.

- Weekly HW, due Tue before class. LaTex. Overleaf.
- 2 midterms, 1 Final.

Office Hours: Tue : 11.30 – 1 Innovation
Thu : 4.15 – 5.45. E450.

## Chapter 2

**Definition** vector space (over $\mathbb{R}$) → set of scalars.

Set $V$ and two operations $+$ and $\cdot$.

$+$ → vector addition or scalar addition.

$\cdot$ → scalar multiplication.

such that

(1) $V$ is closed under vector addition.
$$\vec{v} + \vec{w} \in V \quad \forall \ \vec{v}, \vec{w} \in V$$

(2) $+$ is commutative.

(3) $+$ is associative.

(4) Zero vector $\vec{0} \in V$ s.t. $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$ $\forall \ \vec{v} \in V$.

(5) Additive inverses: for each $\vec{v} \in V$

$\exists \ \vec{-v} \in V$ such that $\vec{v} + \vec{-v} = \vec{0}$

$\exists\, \vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{0}$.

(6) $V$ is closed under scalar multiplication:

$$r \cdot \vec{v} \in V \quad \text{for all } r \in \mathbb{R},\ \vec{v} \in V.$$

(7) scalar multiplication distributes over scalar addition: $(r+s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$

(8) and over vector addition:

$$r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}.$$

(9) multiplication of scalars associates over scalar multiplication: $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$.

(10) multiplication by $1 \in \mathbb{R}$ is the identity operation: $1 \cdot \vec{v} = \vec{v}$.

## Examples

* $\mathbb{R}^n$ n-dimensional vectors.

e.g. $\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\}$

Addition: $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$

Scalar multiplication: $r \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$.

* $L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 3x \right\}$

$= \left\{ \begin{pmatrix} x \\ 3x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$

$= \left\{ x \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$.

* $L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 2x + 1 \right\}$.

$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \notin L \Rightarrow$ L is not a vector space.

(it does not have any 0-vector)

* $\mathcal{P}_n = \{ a_0 + a_1 x + \dots + a_n x^n \mid a_0, \dots, a_n \in \mathbb{R} \}$.

* Matrices: $\mathcal{M}_{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix}$

$$r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}.$$

* Functions

$$\{ f \mid f : \mathbb{R} \to \mathbb{R} \}.$$

Addition: $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

Scalar mult: $(r \cdot f)(x) = r f(x)$.

For example: - $\{ f \mid f : \mathbb{R} \to \mathbb{R},\ f(0) = 0 \}$.

(Check that this a vector space)

- $\{ f \mid f : \mathbb{R} \to \mathbb{R},\ f(0) = 1 \}$.

Not a vector space!

(no zero, not closed under addition or scalar multiplication).

Lemma 16: For any $\vec{v} \in V$ and $r \in \mathbb{R}$ we have

(1) $0 \cdot \vec{v} = \vec{0}$

(2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$

(3) $r \cdot \vec{0} = \vec{0}$.

Two.I.2.

**Definition** For any vector space a subspace is a subset that is itself a vector space (under the inherited operations)

Examples: * $\{\vec{0}\}$ $\rightarrow$ the trivial subspace. and $V$ itself are always subspaces of $V$.

* The x-axis is a subspace of $\mathbb{R}^2$.

$$\left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

* $$\mathcal{P}_3 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 .. a_3 \in \mathbb{R} \}.$$

$$\mathcal{P}_2 = \{ a_0 + a_1 x + a_2 x^2 \mid a_0 ... a_2 \in \mathbb{R} \}$$

$\mathcal{P}_2$ is a subspace of $\mathcal{P}_3$.

$$\{ a_0 + a_3 x^3 \mid a_0, a_3 \in \mathbb{R} \}$$

is also a subspace of $\mathcal{P}_3$.

Definition: a linear combination is an expression of the form

$$r_1 \cdot \vec{v}_1 + r_2 \cdot \vec{v}_2 + \ldots + r_n \cdot \vec{v}_n .$$

Lemma 2.9 The following are equivalent statements:

(1) $S$ is a subspace of $V$

(2) $S$ is closed under linear combinations of any pair of vectors in $S$:

$$r_1 \cdot \vec{s}_1 + r_2 \cdot \vec{s}_2 \in S \quad \forall\, r_1, r_2 \in \mathbb{R}, \ \vec{s}_1, \vec{s}_2 \in S.$$

→ implies closed under vector addition and scalar multiplication.
If we set $\vec{s}_1 = \vec{s}_2$ and $r_1 = 1$ $r_2 = -1$ then we see that $S$ must contain $\vec{0}$.

## Lecture 2 9/2

Example $S = \left\{ \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$

$S \subseteq \mathbb{R}^3.$

Example $\begin{pmatrix}1\\1\\2\end{pmatrix} + 2\cdot\begin{pmatrix}0\\3\\3\end{pmatrix} = \begin{pmatrix}1\\7\\8\end{pmatrix}$

$\in S \qquad \in S \qquad \in S$

To check Lemma 2.9 in general:

$$\begin{pmatrix}a\\b\\a+b\end{pmatrix} + \begin{pmatrix}c\\d\\c+d\end{pmatrix} = \begin{pmatrix}a+c\\b+d\\a+b+c+d\end{pmatrix} = \begin{pmatrix}a+c\\b+d\\(a+c)+(b+d)\end{pmatrix}.$$

$$r\cdot\begin{pmatrix}a\\b\\a+b\end{pmatrix} = \begin{pmatrix}ra\\rb\\r(a+b)\end{pmatrix} = \begin{pmatrix}ra\\rb\\ra+rb\end{pmatrix}. \quad r \in \mathbb{R}.$$

Example $S = \left\{ \begin{pmatrix}a\\b\\a+b+1\end{pmatrix} \;\middle|\; a,b \in \mathbb{R} \right\}.$

scalar multiplication $2\cdot\begin{pmatrix}a\\b\\a+b+1\end{pmatrix} = \begin{pmatrix}2a\\2b\\2a+2b+2\end{pmatrix}$

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$\left\{ \begin{pmatrix}a\\b\end{pmatrix} \;\middle|\; a+b=0 \quad a,b \in \mathbb{R} \right\}.$ is a vector space

$\left\{ \begin{pmatrix}a\\b\end{pmatrix} \;\middle|\; a+b=1 \quad a,b \in \mathbb{R} \right\}$ is not.

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S For a set of vectors $S = \{\vec{v}_1 \quad \vec{v}_n\}$

Span For a set of vectors $S = \{\vec{v}_1 \ldots \vec{v}_n\}$.

we define the span as the set

$$[S] = \{ a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n \mid a_1, a_2 \ldots a_n \in \mathbb{R} \}.$$

- Is the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Yes: $2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

- Is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the span of $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

  Need $a_1, a_2 \in \mathbb{R}$. such that

$$a_1 \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$\Longleftrightarrow$$

$$a_1 \cdot 2 + a_2 \cdot 0 = 2$$
$$a_1 \cdot 2 + a_2 \cdot 3 = 3.$$

Note that if $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_n\} = S \subseteq \mathbb{R}^n$.

then $[S]$ is a subspace.

Two.III Linear independence.

We say that a set of vectors $\vec{v}_1 \dots \vec{v}_n$ is linearly independent if none of them is a linear combination of the others.

Another way of saying this is that the equation $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0}$.

only has one (trivial solution).

→ meaning $a_1 = a_2 = a_3 = \dots = a_n = 0$.

If $v_i = a_1\vec{v}_1 + \dots + a_{i-1}\vec{v}_{i-1} + a_{i+1}\vec{v}_{i+1} + \dots + a_n\vec{v}_n$

Then $a_1\vec{v}_1 + \dots + -\vec{v}_i + \dots a_n\vec{v}_n = \vec{0}$.

We call this a dependence. and then we say the set is dependent.

**Corollary 1.14** A set $S$ is linearly independent iff the removal of any $\vec{v}$ from $S$ shrinks the span: $[S \setminus \{\vec{v}\}] \subsetneq [S] \ \forall \vec{v} \in S$.

"if and only if"

**Corollary 1.17** In a vector space, any finite set of vectors has a linearly independent subset with the same span.

**Two.III** A linearly independent set of vectors that span a vector space $V$ is called a **basis** for $V$.

Example: $\mathbb{R}^n$ we call the set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

the Standard basis for $\mathbb{R}^n$.

**Theorem 1.12** A subset of a vector space is a basis for that space iff every vector in the space can be expressed as a linear combination of the subset in exactly one way.

Suppose that $\vec{v}_1 \dots \vec{v}_n$ form a basis for vector space $V$, and suppose that $\vec{w}$ can be expressed in two ways as a linear combination of $\vec{v}_1 \dots \vec{v}_n$

$$\vec{w} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$$

$$\vec{w} = b_1 \vec{v}_1 + \dots + b_n \vec{v}_n$$

$$\vec{w} - \vec{w} = (a_1 - b_1)\vec{v}_1 + \dots + (a_n - b_n)\vec{v}_n = \vec{0}$$

There should only be the trivial solution to this equation $\Rightarrow a_i - b_i = 0 \quad \forall i$.

$\Rightarrow$ These are not distinct linear combinations.

(we will work only with finite dimensional spaces)

<u>Definition</u> The dimension of a vector space $V$, denoted $\dim(V)$, is the number of elements in a basis for $V$.

## Lecture 3 8/7

Two.III.2.

**Lemma 2.3** (Exchange lemma)

Suppose $\mathcal{B} = (\vec{\beta}_1, \vec{\beta}_2, \dots, \vec{\beta}_n)$ is a basis for a vector space $V$, and

$\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \dots + c_n\vec{\beta}_n$ such that
$\in V$

$c_i \neq 0.$

Then replacing $\vec{\beta}_i$ by $\vec{v}$ gives a new basis for $V$. → call this new basis $\hat{\vec{B}}$.

Proof: Need that $\hat{\vec{B}}$ is linearly independent and it spans $V$.

* Linear independence: Suppose not. Then we have a nontrivial relation

$$d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \dots + d_i\vec{v} + \dots + d_n\vec{\beta}_n = 0.$$

must have $d_i \neq 0$ otherwise this counts as a relation in $B$.

$$\Rightarrow d_1\vec{\beta}_1 + \dots + d_i(c_1\vec{\beta}_1 + \dots + \underbrace{c_i}_{c_i\neq 0.}\vec{\beta}_i + \dots + c_n\vec{\beta}_n) + \dots + d_n\vec{\beta}_n = 0.$$

$$\Rightarrow (d_1 + d_ic_1)\vec{\beta}_1 + \dots + \underbrace{d_ic_i}_{\text{still non-trivial}}\vec{\beta}_i + \dots + (d_n + d_ic_n)\vec{\beta}_n = 0.$$

This gives us a non-trivial relation in $B$. ✗

... relation in $B$. #

✱ $[B] = [\hat{B}]$. → To prove that $S = T$ for two sets: - $S \subseteq T$ - $T \subseteq S$.

↙ Contradiction.

• $[\hat{B}] \subseteq [B]$: If $\vec{w} \in [\hat{B}]$ then

$$\vec{w} = d_1\vec{\beta}_1 + \dots + d_i\vec{v} + \dots + d_n\vec{\beta}_n$$

$$= d_1\vec{\beta}_1 + \dots + d_i(c_1\vec{\beta}_1 + \underbrace{\dots}_{c_i\vec{\beta}_i} + c_n\vec{\beta}_n) + \dots + d_n\vec{\beta}_n$$

$$= (d_1 + d_i c_1)\vec{\beta}_1 + \dots + d_i c_i \vec{\beta}_i + \dots + (d_n + d_i c_n)\vec{\beta}_n.$$

$$\Rightarrow \vec{w} \in [B].$$

Want to show $S \subseteq T$ for two sets $S, T$.


Do this by showing that $a \in S$ implies $a \in T$.

• $[B] \subseteq [\hat{B}]$: Suppose $\vec{w} \in [B]$.

→ these are new $d_1 \dots d_n$

$$\vec{w} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \dots d_i\vec{\beta}_i + \dots + d_n\vec{\beta}_n$$

$$\vec{v} = c_1\vec{\beta}_1 + \dots + c_i\vec{\beta}_i + \dots + c_n\vec{\beta}_n$$ → same $c_1 \dots c_n$

$-c_i \vec{\beta}_i = c_1\vec{\beta}_1 + \dots - \vec{v} + \dots + c_n \vec{\beta}_n$

$\vec{\beta}_i = -\frac{c_1}{c_i}\vec{\beta}_1 + \dots + \frac{1}{c_i}\vec{v} + \dots - \frac{c_n}{c_i}\vec{\beta}_n \qquad (\vec{\beta}_i \in [\hat{B}])$

$\Rightarrow \vec{w} = d_1\vec{\beta}_1 + \dots + d_i\left(-\frac{c_1}{c_i}\vec{\beta}_1 \dots + \frac{1}{c_i}\vec{v} + \dots - \frac{c_n}{c_i}\vec{\beta}_n\right)$

$+ \dots + d_n \vec{\beta}_n.$ remember that $c_i \neq 0$.

$= \left(d_1 - d_i \frac{c_1}{c_i}\right)\vec{\beta}_1 +$ etc. (all in terms of elements of $\hat{B} = \{\vec{\beta}_1, \dots \vec{v}, \dots \vec{\beta}_n\}$ not $\vec{\beta}_i$).

$\Rightarrow \vec{w}$ can be expressed (annoyingly) as a linear combination of elements of $\hat{B}$. $\Rightarrow \vec{w} \in [\hat{B}]$.

We now have that $\hat{B}$ is linearly independent and span $V = [B]$ and is therefore a basis for $V$. $\square$

<u>Example $\mathbb{R}^2$</u> $\qquad B = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$

$$\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

$\Rightarrow$ we can replace any element of $B$ by $\vec{v}$.

$\hat{B}$ $\to$ $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

$\hat{B}$ $\to$ $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$ also a basis for $\mathbb{R}^2$.

Instead, if $\vec{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

$\left\{ \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$ is a basis.

$\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\}$ is <u>not</u> a basis for $\mathbb{R}^2$.

# Lecture 4 8/9.

**<u>Lemma</u>** (other version exchange lemma)

If $\underbrace{B_1}_{=\{\vec{v}_1, \dots \vec{v}_n\}}$ and $\underbrace{B_2}_{=\{\vec{w}_1, \dots \vec{w}_m\}}$ are two bases for a vector space $V$, then for any $\vec{v}_i \in B_1$ there exists a $\vec{w}_j$ in $B_2$ such that

$B_1 \setminus \{\vec{v}_i\} \cup \{\vec{w}_j\}$

(replace $\vec{v}_i$ by $\vec{w}_j$ in $B_1$)

is also a basis for $V$.

set notation

remove one elt

$a \in S$ from $S$:

$S \setminus \{a\}$

<u>Proof</u> By Lemma 2.3

we need to show that $B_2$

contains a $\vec{w}_j$ such that $\vec{w}_j = \underbrace{a_1\vec{v}_1 + \dots a_i\vec{v}_i + \dots + a_n\vec{v}_n}$

has $a_i \neq 0$.

there is a unique representation of $\vec{w}_j$ in terms of $B_1$.

Suppose (for the sake of contradiction) that there is no such $1 \leq j \leq m$.

Then every vector in $B_2$ can be expressed as a linear combination of vectors in $B_1 \setminus \{\vec{v}_i\}$.

We have $\vec{v}_i \in V$ and therefore

$$\vec{v}_i = b_1\vec{w}_1 + b_2\vec{w}_2 + \dots + b_m \vec{w}_m$$

$$\vec{v}_i = b_1(\underbrace{c_{11}\vec{v}_1 + \dots + c_{1n}\vec{v}_n}_{\vec{w}_1 \text{ expressed in terms}}) + \dots + b_m(c_{m1}\vec{v}_1 + \dots + c_{mn}\vec{v}_n)$$

no $\vec{v}_i$

no $\vec{v}_i$

$1 \quad B_1 \setminus \{\vec{v_i}\}$

This can be rewritten as

$$\vec{v_i} = d_1\vec{v_1} + \dots + d_{i-1}\vec{v_{i-1}} + d_{i+1}\vec{v_{i+1}} + \dots + d_n\vec{v_n}.$$

This implies that $\vec{v_i}$ is dependent on the other elements of $B_1 \Rightarrow B_1$ is not linearly independent. #.

$\Rightarrow$ There must be such a $\vec{w_j}$. $\square$.

Corollary: All bases of $V$ (finite dimensional) must have the same cardinality. number of elements

Proof Suppose $|B_1| < |B_2|$. (for the sake of contradiction).
Then we can start replacing the elements of $B_1$ with elements of $B_2$ until $B_1 \subsetneq B_2$. slight abuse of notation
Any remaining element in $B_2$ can now be written in terms of $B_1$ (but that's also other elements in the set $B_2$).

$\Rightarrow B_2$ is not linearly independent. #. $\square$.

<u>Lemma</u> Any linear independent set in $V$ can be extended to a basis (or: is a subset of some basis).

<u>Proof</u>: Let $S$ be a L.I. set in $V$. If $S$ is not a basis then $\exists\ \vec{v} \in V$ that is not a linear combination of $S$. Then $S \cup \{\vec{v}\}$ is still linearly independent. Can continue adding things to $S$ until it is a basis.

<u>Lemma</u> Any spanning set of $V$ can be reduced to a basis (or: is a superset of a basis, or: contains a basis as a subset).

(always assume finite set)

<u>Proof</u>: Let $S$ be a spanning set of $V$. If $S$ is not already a basis then it contains some $\vec{v}$ that is dependent on the other elements of $S$. Then $S \setminus \{\vec{v}\}$ still spans the space. Can continue until "$S$" is linearly independent. $\square$

A linear transformation is a function $T: \mathbb{R}^n \to \mathbb{R}^m$ such that it can be written as $T(\vec{v}) = A\vec{v}$ where $A$ is an $m \times n$ matrix.

all $a_{ij} \in \mathbb{R}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

Dot product: For two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$

$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ The dot product is given by $\vec{v} \cdot \vec{w} = \underbrace{v_1 w_1 + v_2 w_2 + \dots + v_n w_n}_{\text{scalar.}}$

Dot product is commutative and distributes over vector addition

$\vec{v} \cdot (\vec{w} + \vec{u}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{u}$.

Can think of $A$ as a set of row vectors

$$A\vec{v} = \begin{pmatrix} \overbrace{\longleftarrow \; \vec{r}_1 \; \longrightarrow}^{n} \end{pmatrix} \quad \begin{pmatrix} \vec{r}_1 \cdot \vec{v} \end{pmatrix}$$

$$\begin{pmatrix} \leftarrow & \vec{r}_2 & \rightarrow \\ & \vdots & \\ \leftarrow & \vec{r}_m & \rightarrow \end{pmatrix} \vec{v} = \begin{pmatrix} \vdots \\ \vdots \\ \vec{r}_m \cdot \vec{v} \end{pmatrix}$$

$$A\vec{v} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{c}_1 & \cdots & \vec{c}_n \\ \downarrow & & \downarrow \end{pmatrix} \vec{v} = v_1 \vec{c}_1 + v_2 \vec{c}_2 + \cdots + v_n \vec{c}_n.$$

Result is a vector $\vec{w} \in \mathbb{R}^m$.

Suppose we are given an equation: $A\vec{x} = \vec{w}$ where $\vec{x}$ is unknown.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$$

Then we are looking to solve

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= w_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= w_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= w_n. \end{aligned}$$

Example: $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \end{pmatrix} = \begin{pmatrix} 2 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} x_2 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}.$$

- $2x_1 + x_2 = 2 \longrightarrow 2x_1 + 1 = 2 \quad x_1 = \frac{1}{2}.$

  $0 \cdot x_1 + x_2 = 1 \longrightarrow x_2 = 1$

  1 solution unique

- $$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

  $2x_1 + 4x_2 = 1 \longrightarrow$ no solution.

  $x_1 + 2x_2 = 1$

  $2x_1 + 4x_2 = 2$

  $1 = 2.$

  0 solutions

- $$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

  $2x_1 + 4x_2 = 6$

  $x_1 + 2x_2 = 3$

  $\times 2 \quad 2x_1 + 4x_2 = 6$

  There really is only one equation.

  $x_1 + 2x_2 = 3$

  We can choose $x_1$ freely.

  Let $x_1 = a$ and then $x_2 = \frac{3-a}{2}$.

  gives a solution for any $a \in \mathbb{R}$.

$\infty$ solutions.

Lecture 5 9/14.

The column rank of a matrix A is the dimension of their span.
Similarly, the row rank is the dimension of the span of the rows.

* We will see that these are always equal, so we can just talk about rank (A).
  They need not span the same space!
  ↳ meaning the set of rows and set of columns.

Example * $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ → column space: span$\left(\left\{\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}\right\}\right)$ which is 2-dimensional. $= \mathbb{R}^2$

↳ row space $\text{Span}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}\right) = \mathbb{R}^2$ also 2-dimensional.

* $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$ → column space is the span$\left(\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}\right\}\right)$

which is a 2-dimensional subspace of $\mathbb{R}^3$.

The rows have $\text{span}\left(\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix} \right\}\right)$

can check that this is $\mathbb{R}^2 \Rightarrow$ 2-dimensional.

* $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ The row space is 2-dim, because $\vec{r}_2$ is not in the span of $\vec{r}_1$, but $\vec{r}_3 = \frac{1}{4}\vec{r}_1 + \frac{1}{4}\vec{r}_2$.

The column rank is also 2 ...

(find the dependence if you want).

Gauss-Jordan Elimination / Reduced row echelon form (rref).

Given a system of linear equations, there are operations we can perform that do not change the solution set, but help us find it.

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\begin{aligned} a_1 x + a_2 y + a_3 z &= d_1 \\ b_1 x + b_2 y + b_3 z &= d_2 \\ c_1 x + c_2 y + c_3 z &= d_3 \end{aligned}$$

$x\,y\,z$ are variables

- Can change the order of the equations.
- can multiply any equation by a constant (not zero)
- Can add or subtract one equation from another.

$$A = B \quad \text{then} \quad A = B$$
$$C = D \qquad\qquad C - A = D - B.$$

To simplify writing, we will write the entire system as one matrix, called an augmented matrix.

We write $A\vec{v} = \vec{w}$ as $(A \mid \vec{w})$. → where the goal is to find the set of possible values of $\vec{v}$.

For the previous system we would write

$$\left(\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right).$$

We want to write this matrix in r.r.e.f.:

- The leading (i.e. first non-zero) coefficient of each row is 1.
- A column that contains such a leading 1 has 0s everywhere else.

- The rows are in order of leading coefficients.

This can be achieved by (elementary) row operations:
- swapping rows.
- multiplying a row by a nonzero scalar.
- adding a multiple of one row to another.

<u>Algorithm</u> RREF $n \times m$ matrix

for row $i$ in $1 \dots n$:

let $a_{ij}$ be the leading coefficient of row $i$

divide row $i$ by $a_{ij}$ ($a_{ij}$ becomes 1)

subtract row $i$ $a_{kj}$ times from all other rows $k$, so $a_{kj}$ becomes 0 for all $k \neq i$.

end for

order rows by leading coefficients.

Example: $\left(\begin{array}{ccc|c} 3 & 6 & -1 & 3 \\ 2 & -4 & 3 & 2 \end{array}\right)$

divide $\vec{r}_1$ by 3 $\Rightarrow$ $\left(\begin{array}{ccc|c} 1 & 2 & -\frac{1}{3} & 1 \\ 2 & -4 & 3 & 2 \end{array}\right)$

add $\vec{r}_1$ $-2$ times to $\vec{r}_2$ :

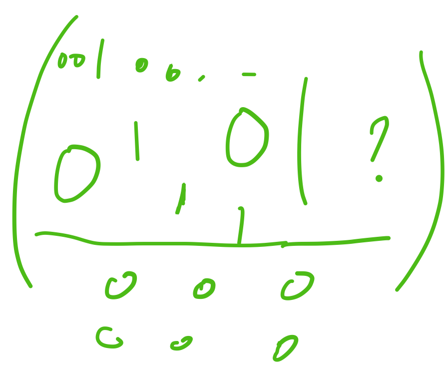
$$\left(\begin{array}{ccc|c} 1 & 2 & -\frac{1}{3} & 1 \\ 0 & -8 & 11/3 & 0 \end{array}\right)$$

multiply $\vec{r}_2$ by $\frac{1}{-8}$ :

$$\left(\begin{array}{ccc|c} 1 & 2 & -\frac{1}{3} & 1 \\ 0 & 1 & -\frac{11}{24} & 0 \end{array}\right)$$

add $\vec{r}_2$ $-2$ times to $\vec{r}_1$ :

$$\left(\begin{array}{ccc|c} 1 & 0 & 7/12 & 1 \\ 0 & 1 & -11/24 & 0 \end{array}\right)$$



$$x + 7/12\, z = 1$$
$$y + \frac{-11}{24} z = 0$$

$z = t \qquad t \in \mathbb{R}$

Then $x = 1 - 7/12\, t$

$y = \frac{11}{24} t$

- $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ We have $0=1$ in the last row.
  - Inconsistent
  - 0 solutions.

- $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$ $x=1$, $y=2$, $z=3$
  - consistent
  - independent
  - 1 solution

- $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $z=t$, $y+2z=0 \Rightarrow y=-2t$
  - Consistent
  - dependent
  - $\infty$ solutions

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad x=0$$

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## Lecture 6 9/16

Proposition / Claim : small facts with simple proofs.

Lemma Lemma : smaller results : usually then used to prove bigger things.

$\Downarrow$ $\Downarrow$

Theorem : ...

Theorems: big, important, useful.

$\Downarrow$

Corollary: follows directly from a theorem, but still worth pointing out.

**Claim:** Row operations do not change the span of the rows.

- Reordering: easy to see.
- Scaling one row (by non zero scalar): fairly straightforward.
- Adding a scalar multiple of one row to another.

Claim: $S = \{\vec{v}_1 \dots \vec{v}_n\}$ Then the set

$$S' = \{\vec{v}_1, \vec{v}_2 + r\cdot\vec{v}_1, \vec{v}_3, \dots, \vec{v}_n\}$$

has the same span: $[S] = [S']$.

Proof: Exercise.

**Claim:** Row operations do not change the <u>dimension</u> of the column span.

Proof. The system $\left(\vec{v}_1 \ \dots \ \vec{v}_n\right) \vec{x} = \vec{\ }$

$$\begin{pmatrix} \vec v_1 & \cdots & \vec v_m \\ \downarrow & & \downarrow \end{pmatrix} \vec x = \vec w .$$

is asking: can we write $\vec w$ as a linear combination of $\vec v_1, \dots, \vec v_m$.

If we write this in rref we still satisfy the same set of linear equations. $\Rightarrow$ we find the same set of solutions.

The space of vectors $\{\vec w \mid \text{there is a solution}\}$ is the same. TBD.

$\Rightarrow$ row rank and column rank are equal.

<u>Proof</u>: We just need to show that they are equal in the rref:

$$\begin{array}{ccccc} \vec v_1 & \vec v_2 & \vec v_3 & s\cdot\vec v_1 + a\cdot\vec v_2 + b\cdot\vec v_3 & \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & s & t \\ 0 & 1 & 0 & a & g \\ 0 & 0 & 1 & b & \cdot \\ 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

row rank: the number of non-zero rows: because each has a leading 1: Cannot be a linear comb. of the others, which have

column rank: also the no. of

also the nr. of 0 in that coordinate.
leading 1s. The columns that have leading 1s. are linearly independent and form a straightforward basis for all other columns after.

## Representations

If we write a vector in $\mathbb{R}^3$ as a linear combination of the standard basis, we get $\vec{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ → recipe to get $\vec{w}$ from the standard basis.

$$\vec{w} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Suppose that we want to use $B = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$ as our basis. Then

$$\mathrm{Rep}_B(\vec{w}) = \begin{pmatrix}x\\y\\z\end{pmatrix}_B$$ means that $\vec{w} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}1\\1\\1\end{pmatrix}$.

We use curly brackets for unordered sets

$\{a, b, c\} = \{b, c, a\}$

and parentheses for ordered $(a,b,c) \neq (a,c,b)$.

For example: Consider $\mathcal{P}_3 = \{a + bx + cx^2 + dx^3 \mid a,b,c,d \in \mathbb{R}\}$.

Natural basis $(1, x, x^2, x^3) = B$

$$\mathrm{Rep}_B(x - x^3) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}_B.$$

Example: find a basis for the set of polynomial $p(x)$ in $\mathcal{P}_3$ such that $p(7) = 0$ and $p(5) = 0$.

$$a + 7b + 49c + 343d = 0$$
$$a + 5b + 25c + 125d = 0$$

Fact: if these are all 0 then the solution set is a vector space.

Augmented matrix:

$$\begin{array}{cccc} a & b & c & d \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 7 & 49 & 343 & 0 \\ 1 & 5 & 25 & 125 & 0 \end{array}\right)$$

R.R.E.F.:

$$\left(\begin{array}{cccc|c} 1 & 0 & -35 & -420 & 0 \end{array}\right.$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 12 & 109 & 0 \end{array}\right)$$

$$\begin{pmatrix} 35 \\ -12 \\ 1 \\ 0 \end{pmatrix}_B \qquad \begin{pmatrix} 420 \\ -109 \\ 0 \\ 1 \end{pmatrix}_B$$