Exercise 1 (3.23 p. 421)

For each, find the characteristic polynomial and the eigenvalues.

(a) $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 3 \\ 7 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

. . . .

Solution.

(a) We have

$$f(\lambda) = \det\begin{pmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{pmatrix} = (10-\lambda)(-2-\lambda) - (-9)\cdot 4 = \lambda^2 - 8\lambda + 16.$$

(b) We have

$$f(\lambda) = \det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 2\cdot 4 = \lambda^2 - 4\lambda - 5.$$

(c) We have

$$f(\lambda) = \det\begin{pmatrix} 0-\lambda & 3 \\ 7 & 0-\lambda \end{pmatrix} = (-\lambda)(-\lambda) - 3\cdot 7 = \lambda^2 - 21.$$

(d) We have

$$f(\lambda) = \det\begin{pmatrix} 0-\lambda & 0 \\ 0 & 0-\lambda \end{pmatrix} = (-\lambda)(-\lambda) = \lambda^2.$$

(e) We have

$$f(\lambda) = \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda) = \lambda^2 - 2\lambda + 1.$$

Exercise 2

We define the **trace** of a matrix as the sum of its diagonal entries: $\text{tr}(A) = a_{11}+a_{22}+\ldots+a_{nn}$. Show that for any 2×2 matrix A, we have

$$f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$$

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Solution. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

Then $\text{tr}(A) = a + d$ and $\det(A) = ad - bc$. We have

$$f_A(\lambda) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \text{tr}(A)\lambda + \det(A).$$

Exercise 3

For which values of x does the following matrix have $\det(A) = \mathrm{tr}(A) = 0$?

$$A = \begin{pmatrix} 1 & 7 & 9 \\ 0 & 1+x & 7 \\ 0 & 0 & x \end{pmatrix}.$$

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Solution. We have $\mathrm{tr}(A) = 1 + (1 + x) + x = 2 + x$. Therefore $\mathrm{tr}(A) = 0$ when $x = -1$. We have $\det(A) = 1 \cdot (1 + x) \cdot x = x^2 + x$. Therefore, we have $\det(A) = 0$ when $x \in \{0, -1\}$. We have both $\det(A) = \mathrm{tr}(A) = 0$ exactly when $x = -1$.

Exercise 4 (3.31 p. 422)

Find the eigenvalues and associated eigenvectors of the matrix representing the differentation operator $d/dx : \mathcal{P}_2 \to \mathcal{P}_2$.

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Solution. Represent $\mathcal{P}_2$ with respect to the standard basis $\langle 1, x, x^2\rangle$. Then d/dx is the map given by $a + bx + cx^2 \mapsto b + 2cx$. Then the matrix representing this map is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix has characteristic equation

$$f_A(\lambda) = \det \begin{pmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 2 \\ 0 & 0 & 0-\lambda \end{pmatrix} = -\lambda^3.$$

This cubic polynomial has only one root: $\lambda = 0$. To find the associated eigenvectors, we have

$$E_0 = \ker \begin{pmatrix} 0-0 & 1 & 0 \\ 0 & 0-0 & 2 \\ 0 & 0 & 0-0 \end{pmatrix} = \ker \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right].$$

Exercise 5

Suppose that a 2×2 matrix A is such that $(A\vec{v}) \cdot \vec{v} = 0$, for all $\vec{v} \in \mathbb{R}^2$. Can A be invertible? What if A is 3×3, and $(A\vec{v}) \cdot \vec{v} = 0$, for all $\vec{v} \in \mathbb{R}^3$?

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Solution. This questions asks whether there is an invertible linear transformation represented by A such that every vector is perpendicular to its image. In $\mathbb{R}^2$, we have seen such an example: a clockwise rotation over $90°$. This clearly has an inverse: a counter-clockwise rotation over $90°$.

In $\mathbb{R}^3$, we have seen that every matrix has at least one real eigenvalue. If this real eigenvalue λ is 0, the matrix is not invertible. If this real eigenvalue λ is not 0, then there is an eigenvector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. Then $(A\vec{v}) \cdot \vec{v} = \lambda||\vec{v}||^2 \neq 0$, since neither λ or the length of $\vec{v}$ are 0. Therefore, there cannot exist such a 3×3 matrix.