Exercise 1 (2.11 p.283)

Perform Gram-Schmidt on the following basis for $\mathbb{R}^2$, and check that the resulting vectors are orthogonal:

$$\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \rangle.$$

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Solution. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. We start by normalizing $\vec{v}_1$:

$$\hat{u}_1 = \frac{\vec{v}_1}{||\vec{v}_1||} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, we find the component of $\vec{v}_2$ that is perpendicular to $\hat{u}_1$:

$$\vec{v}_2^{\perp} = \vec{v}_2 - \text{proj}_{[\hat{u}_1]} = \vec{v}_2 - (\vec{v}_2 \cdot \hat{u}_1)\, \hat{u}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}.$$

Finally, we normalize this vector to be unit length:

$$\hat{u}_2 = \frac{\vec{v}_2}{||\vec{v}_2||} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

To check that $\langle \hat{u}_1, \hat{u}_2 \rangle$ is indeed an orthonormal basis, we check that

$$\hat{u}_1 \cdot \hat{u}_2 = -\frac{1}{2} + \frac{1}{2} = 0.$$

Exercise 2 (3.12 p.292)

Find the projection of the following vector in the subspace:

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad S = \left[\left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \right].$$

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Solution. Using Gram-Schmidt again, we find the following orthonormal basis for S: $\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2}^{-1} \\ 0 \\ \sqrt{2}^{-1} \end{pmatrix} \rangle$. Then, we find the projection

$$\text{proj}_S \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2}^{-1} \\ 0 \\ \sqrt{2}^{-1} \end{pmatrix} \right) \begin{pmatrix} \sqrt{2}^{-1} \\ 0 \\ \sqrt{2}^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{pmatrix}.$$

Exercise 3 (1.1 p.329)

Find the determinant of each of these matrices:

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & -1 \end{pmatrix}.$$

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Solution. Using the method that we learned in class, we find:

$$\det \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} = 3 + 1 = 4,$$
$$\det \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} = 2 \cdot 1 + 1 \cdot 1 = 3,$$
$$\det \begin{pmatrix} 4 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & -1 \end{pmatrix} = 4 \cdot (-3) = -12.$$

Exercise 4 (1.3 p.329)

Verify that the determinant of an upper-triangular 3×3 matrix is the product down the diagonal:

$$\det\left(\begin{smallmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{smallmatrix}\right) = aei.$$

. . . .

Solution. We have

$$\det\left(\begin{smallmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{smallmatrix}\right) = a(ei - 0f) - b(0i - 0f) + c(00 - 0e) = aei.$$