Exercise 1

Show that for an $n \times p$ matrix A and a $p \times m$ matrix B, that if the $\operatorname{im}(B) = \ker(A)$, then $\operatorname{im}(AB) = \{\vec{0}\}$.

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Solution. We have that

$$\begin{aligned}
\operatorname{im}(AB) &= \{AB\vec{v} \mid \vec{v} \in \mathbb{R}^n\} \\
&= \{A\vec{w} \mid \vec{w} \in \operatorname{im}(B)\} \\
&= \{A\vec{w} \mid \vec{w} \in \ker(A)\} \\
&= \{\vec{0}\}.
\end{aligned}$$

Exercise 2

Suppose that V is a subspace of $\mathbb{R}^n$. Prove that any linear transformation $T : V \to V$ can be extended to a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$. In other words, prove that for any such $T(\vec{v})$ there exists an $S(\vec{x})$ (defined for all $\vec{x} \in \mathbb{R}^n$) such that $S(\vec{v}) = T(\vec{v})$ for all $\vec{v} \in V$.

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Solution. We have seen that we can define any linear map in terms of what it does to a basis of the domain. So, if $\langle \vec{\beta}_1, \ldots, \vec{\beta}_p \rangle$ is a basis for V ($\dim V = p$), then the linear map is determined by $T(\vec{\beta}_1), \ldots, T(\vec{\beta}_p)$, since any $\vec{v} \in V$ can be written as $a_1\vec{\beta}_1 + \cdots + a_p\vec{\beta}_p$ and this gives

$$T(\vec{v}) = T(a_1\vec{\beta}_1 + \cdots + a_p\vec{\beta}_p) = a_1T(\vec{\beta}_1) + \cdots + a_pT(\vec{\beta}_p).$$

Now, we can extend the basis $\langle \vec{\beta}_1, \ldots, \vec{\beta}_p \rangle$ to a basis $\langle \vec{\beta}_1, \ldots, \vec{\beta}_p, \vec{\beta}_{p+1}, \ldots, \vec{\beta}_n \rangle$ of $\mathbb{R}^n$. Then, we can let S be a linear transformation of $\mathbb{R}^n$ defined by

$$\begin{aligned}
S(\vec{\beta}_1) &= T(\vec{\beta}_1) \\
&\vdots \\
S(\vec{\beta}_p) &= T(\vec{\beta}_p) \\
S(\vec{\beta}_{p+1}) &= \vec{0} \\
&\vdots \\
S(\vec{\beta}_n) &= \vec{0}.
\end{aligned}$$

This gives, for any $\vec{v} \in V$,

$$S(\vec{v}) = S(a_1\vec{\beta}_1 + \cdots + a_p\vec{\beta}_p) = a_1S(\vec{\beta}_1) + \cdots + a_pS(\vec{\beta}_p) = a_1T(\vec{\beta}_1) + \cdots + a_pT(\vec{\beta}_p) = T(\vec{v}).$$

Note that the choice above for $S(\vec{\beta}_{p+1}), \ldots, S(\vec{\beta}_n)$ was arbitrary: anything works.