Exercise 1 (2.26 p.209)

In general, what is the null space of the differentation transformation $d/dx : \mathcal{P}_n \to \mathcal{P}_n$? What is the null space of the second derivative? The kth derivative?

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Solution. We have that $d/dx : \mathcal{P}_n \to \mathcal{P}_n$ is given by

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \mapsto a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}.$$

Thefore, the null space is given by all elements $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathcal{P}_n$ such that $a_1 = a_2 = \cdots = a_n = 0$. In other words, the null space is the set of constant functions of the form $p(x) = a_0$. This the set $\mathcal{P}_0$.

In general, we have that $d^k/dx^k : \mathcal{P}_n \to \mathcal{P}_n$ is given by

$$a_0+a_1x+a_2x^2+\cdots+a_nx^n \mapsto (k(k-1)\dots1)a_k+((k+1)k\cdot2)a_{k+1}x+\cdots+(n(n-1)\dots(n-k+1))a_nx^k.$$

Thefore, the null space is given by all elements $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathcal{P}_n$ such that $a_k = a_{k+1} = \cdots = a_n = 0$. In other words, the null space is the set of functions of the form $p(x) = a_0 + a_1x + \cdots + a_{k-1}x^{k-1}$. This the set $\mathcal{P}_{k-1}$.

Exercise 2 (2.30 p.209)

For the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f\left(\binom{x}{y}\right) = 2x + y$ sketch the inverse image sets $f^{-1}(-3)$, $f^{-1}(0)$ and $f^{-1}(1)$.

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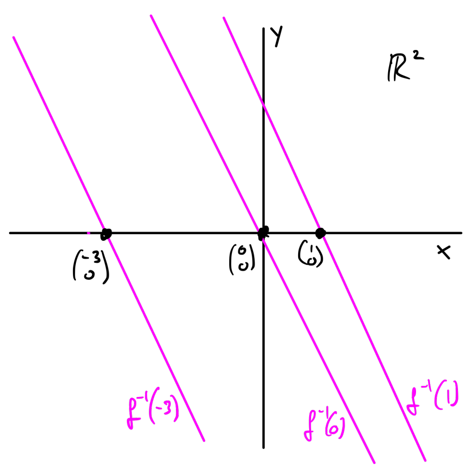
Solution. The inverse image set $f^{-1}(a)$ for the map f is the set of all vectors $\binom{x}{y}$ such that $2x + y = a$. These are the lines of the form $y = -2x + a$. Therefore, we can sketch:



Exercise 3 (2.37 p.209)

Show that a linear map is one-to-one if and only if it preserves linear independence.

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Solution.

"⇒" Let $f : V \to W$ be a linear map and suppose that f is one-to-one. Let $(\vec{v}_1, \ldots, \vec{v}_k)$ be a set of linearly independent vectors in V, and consider the set $(f(\vec{v}_1), \ldots, f(\vec{v}_k))$ in W. Suppose (for the sake of contradiction) that this set is linearly dependent, then we have a nontrivial relation

$$a_1 f(\vec{v}_1) + \cdots + a_k f(\vec{v}_k) = \vec{0}.$$

This implies that

$$f(a_1\vec{v}_1 + \cdots + a_k\vec{v}_k) = a_1 f(\vec{v}_1) + \cdots + a_k f(\vec{v}_k) = \vec{0}.$$

Since is linearly independent we have $\vec{v} = a_1\vec{v}_1 + \cdots + a_k\vec{v}_k \neq \vec{0}$. However, $f(\vec{v}) = f(\vec{0})$, which contradicts f being one-to-one. Therefore, it is not possible that the set $(f(\vec{v}_1), \ldots, f(\vec{v}_k))$ is linearly dependent.

"⇐" Let $f : V \to W$ be a linear map and suppose that f preserves linear independence. Suppose (for the sake of contradiction) that f is not one-to-one. Then we have, for some $\vec{v} \neq \vec{u}$ that $f(\vec{v}) = f(\vec{u})$. Then $f(\vec{v} - \vec{u}) = \vec{0}$ while $\vec{v} - \vec{u} \neq \vec{0}$. However the set $(\vec{v} - \vec{u})$ is linearly independent while $(f(\vec{v} - \vec{u}) = \vec{0})$ is not, which contradicts our assumption that f preserves linear independence. Therefore f must be one-to-one.

Exercise 4 (1.17 p.220)

For a homomorphism from $\mathcal{P}_2$ to $\mathcal{P}_3$ that sends

$$1 \mapsto 1 + x,\ x \mapsto 1 + 2x,\ x^2 \mapsto x - x^3,$$

where does $1 - 3x + 2x^2$ go? Represent this linear map as a matrix with respect to the standard bases for $\mathcal{P}_2$ and $\mathcal{P}_3$.

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Solution. Call this homomorphism f. Since homomorphisms preserve linear combinations we have

$$f(1 - 3x + 2x^2) = 1(1 + x) - 3(1 + 2x) + 2(x - x^3) = -2 - 3x - 2x^3.$$

In terms of standard bases $B = \langle 1, x, x^2 \rangle$ for $\mathcal{P}_2$ and $D = \langle 1, x, x^2, x^3 \rangle$ for $\mathcal{P}_3$, we have that

$$f(1) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}_D,\quad f(x) = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}_D,\quad f(x^2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}_D.$$

This gives us the matrix

$$\text{Rep}_{B,D}(f) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{B,D}.$$

Exercise 5 (1.20 p.220)

Represent the homomorphism $h : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x+z \end{pmatrix}$ with respect to these bases:

$$B = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle, \quad D = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle.$$

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Solution. We have

$$f\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_D$$
$$f\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}_D$$
$$f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}_D$$

This gives us the matrix

$$\text{Rep}_{B,D}(f) = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1/2 & 1/2 \end{pmatrix}_{B,D}.$$

Exercise 6 (1.26 p.221)

Consider the reflection map $f_l : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects all vectors across a line l through the origin. Express this transformation as a matrix with respect to the standard basis. Note that we can define the line l by an angle $0 \leq \theta < \pi$.

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Solution. We look at the images of the standard basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Geometrically, we see that

$$f_l\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}, \quad f_l\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} \sin 2\theta \\ -\cos 2\theta \end{pmatrix}.$$

This gives us the matrix

$$\text{Rep}_{E,E}(f_l) = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Geometric sketch: