Exercise 1 (1.19 p.196)

Are each of the following maps $f : \mathcal{M}_{2\times 2} \to \mathbb{R}$ linear? Justify your answers.

(a) $f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = a + d$,

(b) $f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = ad - bc$,

(c) $f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = a + b + c + d + 1$.

. . . .

Solution.

(a) Yes, this map is linear, since, for any $r, s \in \mathbb{R}$, we have

$$\begin{aligned} f\left(r\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) + s\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\right) &= f\left(\left(\begin{smallmatrix} ra+sa' & rb+sb' \\ rc+sc' & rd+sd' \end{smallmatrix}\right)\right) \\ &= (ra + sa') + (rd + sd') = r(a + d) + s(a' + d') \\ &= rf\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) + sf\left(\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)\right). \end{aligned}$$

(b) This map is not linear. For example, it does not respect scalar multiplication:

$$f\left(2 \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = f\left(\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\right) = 4,$$

while

$$2 \cdot f\left(\cdot\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = 2.$$

(c) This map is not linear. For example, it does not respect scalar multiplication:

$$f\left(2 \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = f\left(\left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = 3,$$

while

$$2 \cdot f\left(\cdot\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = 4.$$

Exercise 2 (1.20 p.196)

We have seen that $f : \mathcal{P}_3 \to \mathcal{P}_2$ given by $f(p(x)) = \frac{dp(x)}{dx}$ is a linear map. What about the map $f : \mathcal{P}_2 \to \mathcal{P}_3$ given by the indefinite integral (where we set the constant to 0), i.e. $f(a + bx + cx^2) = ax + (b/2)x^2 + (c/3)x^3$?

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Solution. For two elements $p(x), q(x) \in \mathcal{P}_2$ and $r, s]in\mathbb{R}$, we have

$$\begin{aligned} f(rp(x) + sq(x)) &= f(r(a + bx + cx^2) + s(a' + b'x + c'x^2)) \\ &= f((ra + sa') + (rb + sb')x + (rc + sc')x^2) \\ &= (ra + sa')x + \frac{rb + sb'}{2}x^2 + \frac{rc + sc'}{3}x^3 \\ &= r(ax + (b/2)x^2 + (c/3)x^3) + s(a'x + (b'/2)x^2 + (c'/3)x^3) \\ &= rf(a + bx + cx^2) + sf(a' + b'x + c'x^2) \\ &= rf(p(x)) + sf(q(x)). \end{aligned}$$

Therefore, f is indeed a homomorphism.

Exercise 3 (1.26 p.197)

Part of the definition of a linear function is that it respects addition. Does a linear function respect subtraction?

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Solution. Let $h : V \to W$ be a linear function, and let $\vec{v}, \vec{u} \in V$. Then

$$h(\vec{v} - \vec{u}) = h(1 \cdot \vec{v} + (-1) \cdot \vec{u}) = 1 \cdot h(\vec{v}) + (-1) \cdot h(\vec{u}) = h(\vec{v}) - h(\vec{u}),$$

since subtraction is a special case of linear combinations. Therefore, linear functions respect subtraction.

Exercise 4 (1.27 p.197)

Assume h is a linear transformation of V and that $(\vec{\beta}_1, \ldots, \vec{\beta}_n)$ is a basis of V. Prove the following statements.

(a) If $h(\vec{\beta}_i) = \vec{0}$ for each basis vector then h is the zero map.

(b) If $h(\vec{\beta}_i) = \vec{\beta}_i$ then h is the identity map.

. . . .

Solution.

(a) For any $\vec{v} \in V$, we have $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Then

$$h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) = c_1\vec{0} + \cdots + c_n\vec{0} = \vec{0}.$$

Therefore, h is the zero map.

(b) For any $\vec{v} \in V$, we have $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Then

$$h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n = \vec{v}.$$

Therefore, h is the identity map.

Exercise 5 (1.32 p.198)

Show that every homomorphism from $\mathbb{R}^1$ to $\mathbb{R}^1$ acts via multiplication by a scalar.

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Solution. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a homomorphism, and let $a = \phi(1)$. Then for any $r \in \mathbb{R}$ we have $\phi(r) = \phi(r \cdot 1) = r \cdot \phi(1) = ra$. Therefore, $\phi(r) = ra$ for all $r \in \mathbb{R}$, and ϕ is a scaling by a a. (Even if $a = 0$ in which case ϕ maps all of $\mathbb{R}$ to 0.)

Exercise 6 (2.21 p.208)

Let $h : \mathcal{P}_3 \to \mathcal{P}_4$ be given by $p(x) \mapsto xp(x)$. Give the range space and null space.

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Solution. Since every $p(x) \in \mathcal{P}_3$ is of the form $p(x) = a + bx + cx^2 + dx^3$ we have that every $h(p(x))$ is of the form $h(p(x)) = ax + bx^2 + cx^3 + dx^4$. Therefore, we have

$$\mathcal{R}(h) = \langle x, x^2, x^3, x^4 \rangle.$$

The only solution to $xp(x) = 0$ must then be $p(x) = 0$, since it implies $a = b = c = d = 0$. Therefore, the null space is the trivial subspace $\{0\}$.