## Exercise 1

Show that if $U, W$ are subspaces of a vector space $V$, then $\dim U \cap W \geq \dim U + \dim W - \dim V$.

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**Solution.** There might be other, easier solutions to this, but here I'll write out the solutions that I hinted at on Teams which uses the Exchange Lemma as it's stated in the book on page 130.

We let $\dim V = n$, $\dim U = p$ and $\dim W = q$. Let $\vec{u}_1, \ldots, \vec{u} + p$ be a basis for $U$. If there is a vector in $W$ and not in $U$, let this be $\vec{w}_1$ (otherwise we are done). From then on, while we have the set $\vec{u}_1, \ldots, \vec{u} + p, \vec{w}_1, \ldots, \vec{w}_i$, find $\vec{w}_{i+1}$ in $W$ such that it is linearly independent of that entire set. At some point, this is no longer possible. Let $\vec{w}_t$ be the last such vector we find. Then the set

$$S = (\vec{u}_1, \ldots, \vec{u} + p, \vec{w}_1, \ldots, \vec{w}_t)$$

is linearly independent in $V$ and therefore $p + t \leq n$. If $t = q$, we are done (since $p + q \leq n$). Otherwise, we continue to complete the linearly independent set $\vec{w}_1, \ldots, \vec{w}_t$ to a basis of $W$. Let $\vec{w}_{t+1}$ be a vector in $W$ that is linearly independent of $\vec{w}_1, \ldots, \vec{w}_t$. Since it is not linearly independent of $S$, we have that

$$\vec{w}_{t+1} = a_1\vec{u}_1 + \cdots + a_p\vec{u}_p + b_1\vec{w}_1 + \cdots + b_t\vec{w}_t.$$

Since $\vec{w}_{t+1}$ is linearly independent of $\vec{w}_1, \ldots, \vec{w}_t$, we must have

$$\vec{w}^*_{t_1} = a_1\vec{u}_1 + \cdots + a_p\vec{u}_p \neq \vec{0}.$$

Then, we can also write

$$\vec{w}^*_{t_1} = \vec{w}_{t+1} - (b_1\vec{w}_1 + \cdots + b_t\vec{w}_t)$$

and by the Exchange Lemma we can exchange $\vec{w}_{t+1}$ for $\vec{w}^*_{t_1}$ in a basis for $W$. Note that $\vec{w}^*_{t_1} \in U \cap W$. We continue in the manner until we have a complete basis for $W$, which looks like

$$\vec{w}_1, \ldots, \vec{w}_t, \vec{w}^*_{t+1}, \ldots, \vec{w}^*_q.$$

In particular, the set $\vec{w}^*_{t+1}, \ldots, \vec{w}^*_q$ is linearly independent in $U \cap W$, and therefore

$$\dim U \cap W \geq q - t \geq q - (n - p) = p + q - n = \dim U + \dim W - \dim V.$$

Compare this question to the following: If I have a class of 10 students, and I form two groups of 6 students each, then you know that there must be at least 2 people in both groups. The groups cannot be disjoint. Something similar happens with dimensions, but of course they are a bit more complicated to work with.

## Exercise 2 (2.21 p134)

Find a basis for the space of cubic polynomials $p(x) \in \mathcal{P}_3$ such that $p(7) = 0$ and $p(5) = 0$.

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**Solution.** We need to find restrictions on $a, b, c, d$ to ensure that polynomials of the form

$$p(x) = a + bx + cx^2 + dx^3$$

have the property $p(5) = p(7) = 0$. We did not ask you to check explicitly that this is indeed a subspace of $\mathcal{P}_3$, but it is. (It would not have been if we had set $p(5), p(7)$ to anything other than 0.) We have the following system of equations that must be satisfied:

$$\begin{aligned} a + 5b + 25c + 125d &= 0 \\ a + 7b + 49c + 343d &= 0 \end{aligned}$$

We find that

$$RREF\left(\begin{array}{cccc|c} 1 & 5 & 25 & 125 & 0 \\ 1 & 7 & 49 & 343 & 0 \end{array}\right) = RREF\left(\begin{array}{cccc|c} 1 & 0 & -35 & -420 & 0 \\ 0 & 1 & 12 & 109 & 0 \end{array}\right).$$

To find a solution for this system, we would let $c = s$ and $d = t$, with $s, t \in \mathbb{R}$. Then $a = 35s + 420t$ and $b = -12s - 109t$. To describe this as a basis for our set of polynomials, we see that we can use the set

$$35 - 12x + x^2, 420 - 109x + x^3$$

as a basis.

## Exercise 3 (3.40 p143)

Prove that a linear system has a solution if and only if the system's matrix of coefficients has the same rank as its augmented matrix.

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**Solution.** I'll show two ways to answer this question. From an rref perspective, we know that the system has a solution if and only if there is no leading 1 in the right-most column of the rref of the augmented matrix. (This would be a row corresponding to an equation $0 = 1$.) Since the rank of a matrix is equal to the number of leading 1s in its rref, this is the case if and only if the augmented matrix has a rank greater than the matrix of coefficients.

Secondly, we can think of rank as the dimension of the column span. We have seen that solving the system of equations $A\vec{x} = \vec{w}$ is the same as asking wehther $\vec{w}$ is in the span of the columns of $A$. We have also seen, for example by Corollary 1.3 on page 109, that $\vec{w}$ is in the span of the columns of $A$ if and only if adding $\vec{w}$ to that set increase the dimension of its span.

## Exercise 4 (1.13 p180)

Show that the map $f : \mathcal{P}_1 \to \mathbb{R}^2$ given by

$$a + bx \mapsto \binom{a-b}{b}$$

is an isomorphism.

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**Solution.** We will first show that this function is a homomorphism. We have

$$\begin{aligned} f(s(a+bx)+t(c+dx)) &= f(sa+tc+sbx+tdx)) \\ &= \binom{sa+tc-sb-td}{sb+td} \\ &= \binom{sa-sb}{sb} + \binom{tc-td}{td} \\ &= s\binom{a-b}{b} + t\binom{c-d}{d} \\ &= sf(a+bx)+tf(c+dx), \end{aligned}$$

as needed. To show that this function is a bijection, we see that is is invertible. If $f(a+bx) = \binom{x}{y}$ we must have that $a = x+y$ and $b = y$.

We can write this system as a matrix transformation as follows:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\binom{a}{b} = \binom{a-b}{b},$$

and the inverse as

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\binom{x}{y} = \binom{x+y}{y}.$$