Exercise 1

Consider three linearly independent vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$. Are the vectors $\vec{v_1}, \vec{v_1}+\vec{v_2}, \vec{v_1}+\vec{v_2}+\vec{v_3}$ also linearly independent?

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Solution. Yes, this set is linearly independent. Suppose that this is not the case. Then, we have a relation of the form

$$a\vec{v_1} + b(\vec{v_1}+\vec{v_2}) + c(\vec{v_1}+\vec{v_2}+\vec{v_3}) = \vec{0}$$

with at least one of the real numbers a, b, c nonzero. Then we can rewrite this as

$$(a+b+c)\vec{v_1} + (b+c)\vec{v_2} + c\vec{v_3} = \vec{0},$$

If $c \neq 0$, then this is a nontrivial relation on the set $\vec{v_1}, \vec{v_2}, \vec{v_3}$, which is a contradiction. Otherwise, if $c = 0$ and $b \neq 0$, we have $b + c \neq 0$, and this gives a nontrivial relation on the set $\vec{v_1}, \vec{v_2}, \vec{v_3}$, which is a contradiction. Finally, if $b = c = 0$ then we must have $a \neq 0$ (since a, b, c are not all zero), which means that $a+b+c \neq 0$, and this also gives a nontrivial relation on the set $\vec{v_1}, \vec{v_2}, \vec{v_3}$, which is a contradiction.

Exercise 2 (2.17)

Find a basis for, and the dimension of, each of the following spaces.

(a) the space

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x - w + z = 0 \right\},$$

(b) the set of 5×5 matrices whose only nonzero entries are on the diagonal,

(c) $\{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + a_1 = 0 \text{ and } a_2 - 2a_3 = 0\} \subseteq \mathcal{P}_3$.

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Solution.

(a) We can rewrite:

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x - w + z = 0 \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$
$$= \left\{ x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Then it is easy to see that the set $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right)$ spans the subspace. Furthermore, these vectors are linearly independent, since they each have a 1 in some position in where all other are 0. Therefore, this subspace is 3-dimensional.

(b) This is the set (the large 0s just mean that everything else off the diagonal is 0)

$$\left\{\begin{pmatrix} a & & & & \\ & b & & & 0 \\ & & c & & \\ 0 & & & d & \\ & & & & e \end{pmatrix} \mid a,b,c,d,e \in \mathbb{R}\right\} =$$

$$\left\{a\begin{pmatrix} 1 & & & & \\ & 0 & & & 0 \\ & & 0 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix} + b\begin{pmatrix} 0 & & & & \\ & 1 & & & 0 \\ & & 0 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix} + c\begin{pmatrix} 0 & & & & \\ & 0 & & & 0 \\ & & 1 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix} + d\begin{pmatrix} 0 & & & & \\ & 0 & & & 0 \\ & & 0 & & \\ 0 & & & 1 & \\ & & & & 0 \end{pmatrix} + e\begin{pmatrix} 0 & & & & \\ & 0 & & & 0 \\ & & 0 & & \\ 0 & & & 0 & \\ & & & & 1 \end{pmatrix} \mid a,b,c,d,e \in \mathbb{R}\right\}$$

This gives us a spanning set

$$\left(\begin{pmatrix} 1 & & & & \\ & 0 & & & 0 \\ & & 0 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 1 & & & 0 \\ & & 0 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 0 & & & 0 \\ & & 1 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 0 & & & 0 \\ & & 0 & & \\ 0 & & & 1 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 0 & & & 0 \\ & & 0 & & \\ 0 & & & 0 & \\ & & & & 1 \end{pmatrix}\right),$$

which again is linearly independent since every element has a 1 in a position where all other elements are 0. Therefore, this is a 5-dimensional subspace of $\mathcal{M}_{5\times 5}$.

(c) We can rewrite

$$\begin{aligned}\{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 + a_1 = 0 \text{ and } a_2 - 2a_3 = 0\} &= \{-a_1 + a_1x + 2a_3x^2 + a_3x^3 \mid a_1, a_3 \in \mathbb{R}\} \\ &= \{a_1(x-1) + a_3(2x^2 + x^3) \mid a_1, a_3 \in \mathbb{R}\}.\end{aligned}$$

Therefore every element can be written as a linear combination of the set $(1-x, 2x^2+x^3)$, which is cleary inearly independent (one is not a scalar mutiple of the other). Therefore this is a 2-dimensiona subspace of $\mathcal{P}_3$.

Exercise 3 (2.20)

Find the dimension of this subspace of $\mathbb{R}^2$:

$$S = \left\{\begin{pmatrix} a+b \\ a+c \end{pmatrix} \mid a,b,c \in \mathbb{R}\right\}.$$

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Solution. Any vector $\binom{x}{y} \in \mathbb{R}^2$ can be written in this form by letting $a = 0,\ b = x,\ c = y$. Therefore $S = \mathbb{R}^2$ and this is a 2-dimensional subspace (the vector space itself).

Exercise 4 (2.37)

Assume U and W are both subspaces of some vector space, and that $U \subseteq W$.

(a) Prove that $\dim(U) \leq \dim(W)$.

(b) Prove that equality of dimension holds if and only if $U = W$.

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Solution.

(a) Let B_U be a basis for U. Then B_U is a linearly independent set in W. If B_U also spans W then we see that $\dim(U) = \dim(W)$. Otherwise, B_U can be extended to a basis B_W of W, by Cor. 2.12 on p.132, which implies that $|B_U| < |B_W|$ and we have $\dim(U) < \dim(W)$. In either case, $\dim(U) \leq \dim(W)$.

(b) We split up the two directions.

"$\Leftarrow$" This direction is easy, but still good to write out explicitly. If $U = W$ then $\dim(U) = \dim(W)$.

"$\Rightarrow$" Suppose that $\dim(U) = \dim(W)$, and for the sake of contradiction suppose that $U \subsetneq W$. Then there is some element $\vec{w} \in W$ and $\vec{w} \notin U$. Let B_U be a basis for U. Since $\vec{w} \notin U$ we have $\vec{w} \notin [B_U]$. This implies that $B_U \cup \{\vec{w}\}$ is a a linearly independent set. By Cor. 2.12 on p.132 this can be extended to a basis of W, which implies that W has dimension at least $|B_U| + 1 = \dim(U) + 1$. This is a contradiction, so therefore we must have $U = W$.

Exercise 5 (3.22)

Give a basis for the column space of this matrix. Give the matrix's rank:

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{pmatrix}.$$

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Solution. There are different ways of phrasing this problem as a system of linear equations. We note that we can look for relations on the column vectors, by solving the system

$$\left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4 & 0 \end{array}\right).$$

In RREF this gives:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 9/4 & 0 \\ 0 & 0 & 1 & 7/4 & 0 \end{array}\right).$$

Therefore, we see that the first three vectors are linearly independent and form a basis for our matrix, which has rank 3. Basis: $\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right)$. Of course, since a 3-dimensional subspace of $\mathbb{R}^3$ is just $\mathbb{R}^3$ (by the result from the previous question), we can also use the standard basis $\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$.

Exercise 6 (3.26)

Given $a, b, c \in \mathbb{R}$, what value of d will cause this matrix to have a rank of 1?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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Solution. This matrix has rank 1 if the column vectors span a 1-dimensional space. In other words, if the second column is a scalar multiple of the first. In that case, $\begin{pmatrix} a \\ c \end{pmatrix} = r\begin{pmatrix} b \\ d \end{pmatrix} = \frac{a}{b}\begin{pmatrix} b \\ d \end{pmatrix}$. This is satisfied if $\frac{a}{b}d = c$, $d = \frac{bc}{a}$.

Exercise 7

If A is a matrix in reduced row echelon form, and a column is removed, is it still in reduced row echelon form? What if a row is removed? Justify your answer carefully.

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Solution. If a column is removed from the matrix, it may no longer be in reduced row echelon form. For example, if we remove a column with a leading 1, then there may be other remaining nonzero values in that row, such that the first nonzero value of the row is no longer 1. Removing a row is not a problem, since it keeps the leading 1s in order (even if one gets removed with it) and leaves a leading 1 in each nonzero row, and it does not introduce any nonzero values in columns that contain a leading 1.

Exercise 8

Read Section Three.IV and give an example of a 2×2 matrix and its inverse (that is not an example from the book).

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Solution. We can invert matrices by hand (although we will learn more about how to do this later on). If, for example, we take the map

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}.$$

Then the system that would invert this map needs to map $\begin{pmatrix} x+y \\ y \end{pmatrix}$ back to $\begin{pmatrix} x \\ y \end{pmatrix}$. If such a matrix is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x+y \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

then $a(x+y) + by = x$ and $c(x+y) + dy = y$. Therefore $a = 1,\ b = -1,\ c = 0,\ d = 1$, and we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$