Carefully justify every answer.

Exercise 1 (Two.I.1.13)

In a vector space every element has an additive inverse. Can some elements have two or more?

....

Solution. An element cannot have more than one inverse. In order to prove this, suppose (for the sake of contradiction) that an element $\vec{v}$ in a vector space V has two distinct inverses $\vec{w}$ and $\vec{u}$. Then $\vec{w} + \vec{v} = \vec{0}$ and $\vec{v} + \vec{u} = \vec{0}$. However, by the fact that vector addition is associative, this implies that

$$\vec{w} = \vec{w} + \vec{0} = \vec{w} + (\vec{v} + \vec{u}) = (\vec{w} + \vec{v}) + \vec{u} = \vec{0} + \vec{u} = \vec{u}.$$

However, this contradicts $\vec{w}$ and $\vec{u}$ being distinct vectors.

Exercise 2 (Two.I.2.44)

Show that if a vector is in the span of a set then adding that vector to the set won't make the span any bigger.

....

Solution. Let $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a set of vectors and let $\vec{w}$ be in the span of S. Then

$$\vec{w} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n.$$

Suppose (for the sake of contradiction) that there is an element $\vec{u}$ in the span of $S \cup \{\vec{w}\}$ that is not in the span of S. Then

$$\vec{u} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n + b\vec{w}.$$

However, we can rewrite this as

$$\begin{aligned}\vec{u} &= b_1\vec{v}_1 + \cdots + b_n\vec{v}_n + b(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n)\\ &= (b_1 + ba_1)\vec{v}_1 + \cdots + (b_n + ba_n)\vec{v}_n,\end{aligned}$$

which implies that $\vec{u}$ can be written as a linear combination of the elements of S, which contradicts out initial assumption. Therefore, the span of S cannot get any bigger by adding $\vec{w}$.

Exercise 3 (Two.I.2.48)

If $S \subseteq T$ are subsets of a vector space, is $[S] \subseteq [T]$?

....

Solution. Yes, this is the case. Suppose that $S \subseteq T$. Let $\vec{v} \in [S]$. Then $\vec{v}$ can be written as a linear combination of the elements of S. However, since $S \subseteq T$, $\vec{v}$ can be written as a linear combination of the elements of T. Therefore, $\vec{v} \in [T]$. This shows that $[S] \subseteq [T]$.

Exercise 4 (Two.III.2.15)

Find a basis for, and the dimension of, $\mathcal{P}_2$. (See Example 1.8 on p. 88.)

. . . .

Solution. We have that

$$\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}.$$

Consider the set $\{1, x, x^2\}$. It is easily verified that none of these elements can be written as a linear combination of the others, so this set is linearly independent. They also span the set $\mathcal{P}_2$, since any element $a_0 + a_1 x + a_2 x^2 = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$ in $\mathcal{P}_2$ is clearly a linear combination of the set $\{1, x, x^2\}$. Therefore, this set forms a basis of $\mathcal{P}_2$, and $\mathcal{P}_2$ is a 3-dimensional vector space over $\mathbb{R}$.