Exercise 1 (2.1.20)

Show that the set of 2×2 matrices with real entries under the usual matrix operations form a vector space.

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Solution. We name this vector space M. We use the definition of a vector space: Definition 1.1 on page 84. There are 10 conditions to check.

(1) **Closure under vector addition.** For any $\vec{v}, \vec{w} \in M$, we have

$$\vec{v} + \vec{w} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \in M,$$

since $a, b, c, d, e, f, g, h \in \mathbb{R}$ and the sum of any two real numbers is also a real number.

(2) **Commutativity of vector addition.**For any $\vec{v}, \vec{w} \in M$, we have

$$\begin{aligned} \vec{v} + \vec{w} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} \\ &= \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \vec{w} + \vec{v}, \end{aligned}$$

since addition of real numbers is commutative.

(3) **Associativity of vector addition.** For any $\vec{v}, \vec{w}, \vec{u} \in M$, we have

$$\begin{aligned} (\vec{v} + \vec{w}) + \vec{u} &= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) + \begin{pmatrix} i & j \\ k & l \end{pmatrix} \\ &= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} (a+e)+i & (b+f)+j \\ (c+g)+k & (d+h)+l \end{pmatrix} \\ &= \begin{pmatrix} a+(e+i) & b+(f+j) \\ c+(g+k) & d+(h+l) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left(\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix} \right) = \vec{v} + (\vec{w} + \vec{u}), \end{aligned}$$

by associativity of addition of real numbers.

(4) **Existence of a zero vector.** We have that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M,$$

and for any $\vec{v} \in M$, we have

$$\vec{v} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \vec{v}.$$

(5) **Additive inverses.** For each $\vec{v} \in M$, we let $\vec{w}$ be its inverse, given by

$$\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

Clearly, $\vec{w} \in M$, and we have

$$\vec{v} + \vec{w} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} a-a & b-b \\ c-c & d-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(6) **Closure under scalar multiplication.** For any $\vec{v} \in M$ and $r\ in \mathbb{R}$, we have

$$r \cdot \vec{v} = r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} \in M,$$

since the real numbers are closed under multiplication.

(7) **Scalar multiplication distributes over scalar addition.** For any $\vec{v} \in M$ and $r, s \in \mathbb{R}$, we have

$$\begin{aligned}(r+s) \cdot \vec{v} = (r+s) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} (r+s)a & (r+s)b \\ (r+s)c & (r+s)d \end{pmatrix} \\ &= \begin{pmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} + \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix} = r \cdot \vec{v} + s \cdot \vec{v},\end{aligned}$$

by the fact that multiplication distributes over addition in the real numbers.

(8) **Scalar multiplication distributes over vector addition.** For any $\vec{v}, \vec{w} \in M$ and $r \in \mathbb{R}$, we have

$$\begin{aligned} r \cdot (\vec{v} + \vec{w}) &= r \cdot \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) = r \cdot \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} r(a+e) & r(b+f) \\ r(c+g) & r(d+h) \end{pmatrix} \\ &= \begin{pmatrix} ra+re & rb+rf \\ rc+rg & rd+rh \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} + \begin{pmatrix} re & rf \\ rg & rh \end{pmatrix} = r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} + r \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &= r \cdot \vec{v} + r \cdot \vec{w}, \end{aligned}$$

by the fact that multiplication distributes over addition in the real numbers.

(9) **Multiplication of scalars associates with scalar multiplication.** For any $\vec{v} \in M$ and $r, s \in \mathbb{R}$, we have

$$(rs) \cdot \vec{v} = (rs) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (rs)a & (rs)b \\ (rs)c & (rs)d \end{pmatrix} = \begin{pmatrix} r(sa) & r(sb) \\ r(sc) & r(sd) \end{pmatrix} = r \cdot \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix} = r \cdot (s \cdot \vec{v}),$$

by associativity of multiplication in the real numbers.

(10) **Multiplication by 1 is the identity operation.** For any $\vec{v} \in M$ we have

$$1 \cdot \vec{v} = 1 \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1a & 1b \\ 1c & 1d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \vec{v}.$$

Exercise 2 (2.1.22)

Show that the following set, under operations inherited from $\mathbb{R}^3$, is not a vector space:

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \right\}.$$

. . . .

Solution. This set is not a vector space, for example, because it does not contain the zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, since $0^2 + 0^2 + 0^2 \neq 1$.

Exercise 3 (2.1.24)

Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

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Solution. The set of rational numbers is not a vector space over the real numbers, since this set is not closed under scalar multiplication. For example, we have $\pi \in \mathbb{R}$ and $1 \in \mathbb{Q}$. Then, if the set was closed under scalar multiplication, we should have $\pi \cdot 1 = \pi \in \mathbb{Q}$, but this is not the case.

Exercise 4

Determine, in each case, whether W is a subspace of $\mathbb{R}^3$. You may use Lemma 2.9 from page 98, or use only items (1), (4), (6) from Definition 1.1 on page 84 and assume that the rest are inherited from $\mathbb{R}^3$.

(a) $W = \left\{ \begin{pmatrix} a \\ a \\ a \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$,

(b) $W = \left\{ \begin{pmatrix} a+1 \\ a+2 \\ a+3 \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$,

(c) $W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R}, \;\; a < b < c \right\}$,

. . . .

Solution.

(a) $W = \left\{ \begin{pmatrix} a \\ a \\ a \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$. This is a vector space over the real numbers. We check condition (2) from Lemma 2.9. Let $\vec{v}, \vec{w} \in W$ and $r, s \in \mathbb{R}$. Then

$$r\vec{v} + s\vec{w} = r \begin{pmatrix} a \\ a \\ a \end{pmatrix} + s \cdot \begin{pmatrix} b \\ b \\ b \end{pmatrix} = \begin{pmatrix} ra \\ ra \\ ra \end{pmatrix} + \begin{pmatrix} sb \\ sb \\ sb \end{pmatrix} = \begin{pmatrix} ra + sb \\ ra + sb \\ ra + sb \end{pmatrix} \in W.$$

(b) $W = \left\{ \begin{pmatrix} a+1 \\ a+2 \\ a+3 \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}$. This is not a vector space over $\mathbb{R}$. For example, it does not contain the zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, since the vector entries cannot be equal.

(c) $W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R},\; a < b < c \right\}$. This is not a vector space over $\mathbb{R}$. For example, it does not contain the zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, since the vector entries cannot be equal.

Exercise 5

Can you find a subset of $\mathbb{R}^2$ that is closed under addition, but not scalar multiplication? Can you find a subset of $\mathbb{R}^2$ that is closed under scalar multiplication, but not addition?

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Solution. There are many possible answers here. See Question 1.45 on page 96, for example.

Exercise 6

For which values of a and b is $\begin{pmatrix} 2 \\ -4 \\ a \\ b \end{pmatrix}$ in the span of the set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \right\}$?

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Solution. We will learn how to solve this more systematically later, but for now we observe that if

$$r \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + s \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ a \\ b \end{pmatrix},$$

then

$$r + 2s = 2 \tag{1}$$
$$r + 3s = -4 \tag{2}$$
$$4s = a \tag{3}$$
$$r + 5s = b. \tag{4}$$

We conclude from (3) that $s = a/4$. By subtracting (1) from (2) we obtain $s = -6$ and therefore $a = -24$, and (1) and (2) are satisfied if $r = 14$. Then by (4) we must have that $b = -16$. Therefore $a = 8, b = -16$ is the unique solution.