How to convert the equations of electromagnetism from Gaussian to SI units in less than no time

Bernard Leroy
Observatoire de Paris, Département d'astrophysique fondamentale, 92195 Meudon Principal Cedex, France

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I. INTRODUCTION

Recently E. A. Desloge[1] presented the theory which lies behind the so-called systems of units in which several fundamental constants ($\hbar$, c, G,...) are set equal to unity. In particular, he showed how these constants can be restored in a systematic way at any stage of a calculation. The restoration technique is so simple that, after having read Desloge's paper, I immediately thought of a somewhat unexpected, but very useful application: the transcription of the equations of electromagnetism from the Gaussian to the MKSA system of units.

The basic idea is this. If one momentarily forgets that the Système International (SI) is rationalized while the Gaussian system is not, one can regard the latter as a "pseudosystem" derived from the SI by setting $\epsilon_0 = \mu_0 = 1$; in much the same way as it is customary in many theoretical developments to consider pseudosystems in which, for example, $\hbar = c = 1$. Then, when the constants ϵ_0 and μ_0 are restored as described in Desloge's paper, equations originally written in Gaussian units assume their familiar SI form (up to a rationalization which is easily performed, as explained below).

The purpose of this paper is to show that the restoration technique enables one to perform this conversion easily. Even the very simple technique of dimensional analysis proposed by Remillard[2] and extended by myself[3] looks complicated in comparison.

II. THE METHOD

As everybody knows, the Gaussian system of units differs from the SI in that (apart from its having fewer basic dimensions) it is not rationalized, and the permittivity ϵ_0 and the permeability μ_0 of free space are taken as dimensionless and equal to unity. As said in the introduction, I shall, for the present purpose, consider the Gaussian system not as system of units, but rather as a *dimensionless* version of the *unrationalized* MKSA system in which it has been chosen to set the *dimensionless* forms of ϵ_0 and μ_0 equal to unity.

From this point of view, the equations written in Gaussian units are to be considered as dimensionless. Specifically, any quantity A', say, appearing in these equations is to be regarded as the dimensionless form of a corresponding quantity A with dimensions $L^{\alpha}M^{\beta}T^{\gamma}I^{\delta}$ in SI units. [As we are interested only in electromagnetic quantities we consider only the four basic dimensions: length (L), mass (M), time (T), and current intensity (I).] This means that

$$A' = A/L^{\alpha}M^{\beta}T^{\gamma}I^{\delta}, \tag{1}$$

where, according to Desloge, L, M, T, I are considered not only as symbols of dimensions but also as references of length, mass, time, and current intensity, respectively. In fact, as the symbols are usually the same in Gaussian and SI units, it is to be understood that, whenever A appears in an expression written in Gaussian units, it is a short-hand notation for A'.

Now, the restoration procedure described by Desloge consists in our case in (i) choosing the values of L, M, T, and I in such a way that ϵ_0 and μ_0, when expressed in dimensionless form, will each assume the value unity; and (ii) plugging theses values in Eq. (1).

As said before, once this has been done, the form then assumed by the equations is that of the unrationalized MKSA system. The transition to the SI is then achieved by means of the following simple "rationalizing" transformations:

$$\begin{aligned} \epsilon_0 &\to 4\pi\epsilon_0, \\ \mu_0 &\to \mu_0/(4\pi), \\ \mathbf{D} &\to 4\pi\mathbf{D}, \\ \mathbf{H} &\to 4\pi\mathbf{H}, \end{aligned} \tag{2}$$

where **D** and **H** are the electric displacement and magnetic field, respectively.

III. EXAMPLES

As we have just said, the first thing to do is to determine the values of L, M, T, and I that make ϵ_0 and μ_0 in dimensionless form equal to unity. The dimensions of the permittivity and permeability are, respectively,

$$\begin{aligned} [\epsilon_0] &= L^{-3}M^{-1}T^4I^2, \\ [\mu_0] &= LMT^{-2}I^{-2}. \end{aligned} \tag{3}$$

The preceding prescription is thus

$$\begin{aligned} \epsilon_0/L^{-3}M^{-1}T^4I^2 &= 1, \\ \mu_0/LMT^{-2}I^{-2} &= 1. \end{aligned} \tag{4}$$

Upon taking the logarithm of these equations, we obtain

$$\begin{aligned} -3\ln L - \ln M + 4\ln T + 2\ln I &= \ln\epsilon_0, \\ \ln L + \ln M - 2\ln T - 2\ln I &= \ln\mu_0. \end{aligned} \tag{5}$$

Dimensional analysis tells us that, since we are working with four basic dimensions, we could have suppressed four and not just two dimensionally independent constants. We are therefore free to impose two more conditions; the following turn out to be most convenient:

$$I = M = 1. \tag{6}$$

Substituting these values into Eqs. (5), these equations are easily solved and we find

$$\begin{aligned} L &= \epsilon_0^{-1}\mu_0^{-2}, \\ T &= \mu_0^{-3/2}\epsilon_0^{-1/2}. \end{aligned} \tag{7}$$

As a check of the consistency of these manipulations one notice that Eqs. (7) imply

$$LT^{-1} = \epsilon_0^{-1/2}\mu_0^{-1/2} = c, \tag{8}$$

where c is the speed of light *in vacuo*.

We are now ready to convert any equation from the

Gaussian system to the SI. Therefore here are some specific examples.

Example 1. Consider Ampère's law in Gaussian units:

$$\nabla\times\mathbf{H}=\left(\frac{4\pi}{c}\right)\mathbf{j}+\left(\frac{1}{c}\right)\frac{\partial\mathbf{D}}{\partial t}, \tag{9}$$

where the symbols have their usual meaning. From the point of view developed here it must be read as

$$\left(\frac{\nabla}{L^{-1}}\right)\times\left(\frac{\mathbf{H}}{IL^{-1}}\right)=4\pi\left(\frac{LT^{-1}}{c}\right)\left(\frac{\mathbf{j}}{IL^{-2}}\right)+\left(\frac{LT^{-1}}{c}\right)\frac{T\partial}{\partial t}\left(\frac{\mathbf{D}}{ITL^{-2}}\right) \tag{10}$$

which, after some simple manipulations taking Eq. (6) into account, gives

$$\nabla\times\mathbf{H}=\frac{4\pi}{c}\mathbf{j}(LT^{-1})+(LT^{-1})\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}. \tag{11}$$

But, as we have seen [cf. Eq. (8)]:

$$LT^{-1}=c,$$

hence Eq. (11) becomes

$$\nabla\times\mathbf{H}=4\pi\mathbf{j}+\frac{\partial\mathbf{D}}{\partial t}. \tag{12}$$

As said earlier, this is the unrationalized form of Ampère's equation in MKSA units; we rationalize it with the help of Eqs. (2), which yield

$$\nabla\times\mathbf{H}=\mathbf{j}+\frac{\partial\mathbf{D}}{\partial t}. \tag{13}$$

This is the well-known form of Ampère's law in SI units.

Example 2. The electric displacement in a dielectric is (in Gaussian units)

$$\mathbf{D}=\mathbf{E}+4\pi\mathbf{P}, \tag{14}$$

where **E** is the electric field and **P** the electric polarization. This is to be read as

$$\left(\frac{\mathbf{D}}{ITL^{-2}}\right)=\left(\frac{\mathbf{E}}{LMI^{-1}T^{-3}}\right)+4\pi\left(\frac{\mathbf{P}}{ITL^{-2}}\right). \tag{15}$$

After some elementary manipulations this becomes [taking Eq. (6) into account]

$$\mathbf{D}=(L^{-3}T^{4})\mathbf{E}+4\pi\mathbf{P}. \tag{16}$$

Now, by virtue of Eqs. (7), we have

$$L^{-3}T^{4}=\epsilon_0.$$

Consequently, Eq. (16) can be written as

$$\mathbf{D}=\epsilon_0\mathbf{E}+4\pi\mathbf{P}, \tag{17}$$

which, after rationalization, gives

$$\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}, \tag{18}$$

the well-known expression of the displacement written in SI units.

Example 3. To lowest order the vector potential of a radiating system in the radiation zone is (in Gaussian units)[4]:

$$\mathbf{A}(\mathbf{x})=c^{-1}r^{-1}e^{ikr}\int\mathbf{J}(\mathbf{x}')d^3x'. \tag{19}$$

As $\mathbf{B}=\nabla\times\mathbf{A}$, the dimensions of the vector potential are $[\mathbf{A}]=L\,[\mathbf{B}]=LMT^{-2}I^{-1}$.

Taking this into consideration, we read Eq. (19) as

$$\frac{\mathbf{A}(\mathbf{x})}{LMT^{-2}I^{-1}}=e^{ikr}\left(\frac{LT^{-1}}{c}\right)\left(\frac{L}{r}\right)\int\left(\frac{\mathbf{J}(\mathbf{x}')}{IL^{-2}}\right)\frac{d^3x'}{L^3} \tag{20}$$

which, in view of Eq. (6), can be written as

$$\mathbf{A}(\mathbf{x})=(L^2T^{-3})(cr)^{-1}e^{ikr}\int\mathbf{J}(\mathbf{x}')d^3x'. \tag{21}$$

But, by virtue of Eqs. (7) and (8), we have

$$L^2T^{-3}/c=\mu_0, \tag{22}$$

and Eq. (21) becomes

$$\mathbf{A}(\mathbf{x})=\mu_0 r^{-1}e^{ikr}\int\mathbf{J}(\mathbf{x}')d^3x'. \tag{23}$$

After rationalization this equation assumes the familiar SI form

$$\mathbf{A}(\mathbf{x})=\left(\frac{\mu_0}{4\pi r}\right)e^{ikr}\int\mathbf{J}(\mathbf{x}')d^3x'. \tag{24}$$

IV. CONCLUSION

Similar examples could have been given *ad infinitum*. However, I believe that these, which were chosen virtually at random, should suffice to convince the reader that the method is extremely simple and very easy to use. Besides, it might be worth stressing that it is in no way limited to the conversion between the Gaussian system and the Système International.

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[3]B. Leroy, Am. J. Phys. **52**, 230 (1984).

[4]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 394.