



Fig. 2. One of the five 3d states with $m = +2$.

are also doughnut shaped.⁸ We must conclude, therefore, P plots are fundamentally incapable of representing these atomic states distinctly. The $\text{Re}(\psi)$ plots, on the other hand, will show real differences: The two 3d states, will each have four blobs, whereas each corresponding 2p state will have two.

If greater accuracy is desired in a picture of $\text{Re}(\psi)$, we can create more elaborate graphics that have multiple-

layered contours. These are, however, matters of technical details that we shall not elaborate on.

Last, an additional remark: Of course, the P plots can still be useful. There are other situations where P as such is of interest, and P plots can be adequate. One example is, in fact, in Ref. 3.

¹R. L. W. Chen, "Computer graphics for solutions of time-dependent Schrödinger equations," *Am. J. Phys.* **50**, 902-906 (1982).

²R. L. W. Chen, "Addendum: Computer graphics for solutions of time-dependent Schrödinger equations: Note on the numerical solutions of the Schrödinger equation," *Am. J. Phys.* **51**, 570 (1983).

³R. L. W. Chen and M. B. Rhode, "Correspondence principle demonstrated," *Am. J. Phys.* **52**, 988-992 (1984).

⁴R. L. W. Chen, "Derivation of the real form of Schrödinger's equation for a nonconservative system and the unique relation between $\text{Re}(\psi)$ and $\text{Im}(\psi)$," *J. Math. Phys.* **30**, 83-86 (1989).

⁵Reference 4, pp. 83-84. Original papers of Schrödinger cited on p. 86.

⁶T. L. Brown and H. E. LeMay, Jr., *Chemistry* (Prentice-Hall, Englewood Cliffs, NJ, 1985), 4th ed. Note in particular the remark on p. 146 (middle of the page): The contour surfaces of the 2p orbitals are shown in Fig. 5.18 (figure is on the opposite page).

⁷D. Halliday and R. Resnick, *Physics* (Wiley, New York, 1986), Pt. II, p. 1194, Fig. 51-14. While figures like those in Ref. 6 are very common, this one is rare.

⁸In Ref. 6, figures of 3d states are shown on the same page as those of 2p.

Laplace's equation with physically uncommon boundary conditions

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Consider the following electrostatic problem.

Problem: Two opposite sides of an infinite cylinder with rectangular cross section of length a and width b are grounded. The other two sides are held at potentials V_1 and V_2 . What is the electrostatic potential Φ at all points inside the cylinder?

Solution: Noting that Φ is independent of z (chosen to be the infinitely long side), we start with the Laplace's equation in two dimensions, $\nabla^2\Phi = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 = 0$, separate the variables, $\Phi(x,y) = X(x)Y(y)$, and turn Laplace's equation into the following two ordinary differential equations:

$$\frac{1}{X} \frac{d^2X}{dx^2} = -\frac{1}{Y} \frac{d^2Y}{dy^2} = k^2. \quad (1)$$

These equations have the general solutions

$$\begin{aligned} X(x) &= A_k e^{kx} + B_k e^{-kx}, \\ Y(y) &= C_k \cos ky + D_k \sin ky \end{aligned} \quad (2)$$

to which we must apply the following boundary conditions (BCs):

$$\begin{aligned} \Phi(0,y) &= V_1, & \Phi(a,y) &= V_2, \\ \Phi(x,0) &= 0, & \Phi(x,b) &= 0. \end{aligned} \quad (3)$$

The BC's on the y variable give

$$C_k = 0, \quad \forall k$$

and

$$\sin kb = 0 \Rightarrow k = n\pi/b, \quad n = 1, 2, 3, \dots$$

and reduce the number of unknown coefficients to two, leading to the following potential function:

$$\Phi(x,y) = \sum_{n=1}^{\infty} (A_n e^{(n\pi/b)x} + B_n e^{-(n\pi/b)x}) \sin\left(\frac{n\pi}{b}y\right). \quad (4)$$

The imposition of the other BCs now determines the remaining two coefficients. We have

$$\begin{aligned} \Phi(0,y) = V_1 &= \sum_{n=1}^{\infty} (A_n + B_n) \sin\left(\frac{n\pi}{b}y\right), \\ \Phi(a,y) = V_2 &= \sum_{n=1}^{\infty} (A_n e^{(n\pi/b)a} + B_n e^{-(n\pi/b)a}) \\ &\quad \times \sin\left[\left(\frac{n\pi}{b}\right)y\right]. \end{aligned}$$

These are Fourier series with inverses

$$A_n + B_n = \frac{2V_1}{b} \int_0^b \sin\left(\frac{n\pi}{b}y\right) dy = \frac{2V_1}{n\pi} [1 - (-1)^n]$$

and

$$\begin{aligned} A_n e^{n\pi a/b} + B_n e^{-n\pi a/b} \\ = \frac{2V_2}{b} \int_0^b \sin\left(\frac{n\pi}{b}y\right) dy = \frac{2V_2}{n\pi} [1 - (-1)^n], \end{aligned}$$

which can be solved to give

$$A_n = (1/n\pi) [1 - (-1)^n] \\ \times [(V_2 - V_1 e^{-n\pi a/b}) / \sinh(n\pi a/b)], \\ B_n = (1/n\pi) [1 - (-1)^n] \\ \times [(V_1 e^{n\pi a/b} - V_2) / \sinh(n\pi a/b)].$$

The problem described above seems simple enough. Nevertheless, a typical EM text does not treat that problem, but a simplified version of it in which either $V_1 = 0$ or $a = \infty$. That is why I found myself in an awkward position in my advanced undergraduate EM class when I tried to "explain" why the atypical problem in which *all* four sides were held at different potentials did not submit itself to the foregoing procedure. Specifically, I decided to choose the boundary conditions

$$\Phi(0,y) = V_1, \quad \Phi(a,y) = V_2, \\ \Phi(x,0) = 0, \quad \Phi(x,b) = V_0. \quad (5)$$

Using Eq. (2) and the third BC in (5) gives $C_k = 0$, leading to the "most general" potential function

$$\Phi(x,y) = \sum_k (A_k e^{kx} + B_k e^{-kx}) \sin ky,$$

which refuses to accept the last BC!

It took me a long time to realize that the k in Eq. (1) *can* in fact be zero, and that it is precisely *this* choice that solves the problem! This point is neither emphasized nor made clear in the standard pedagogical EM literature. I should immediately point out that there are several excellent books treating a variety of boundary value problems. Some of them even *mention* the fact that the $k = 0$ term exists.¹ However, the importance of this term in solving the problem stated above is uniformly overlooked. In my subsequent search I looked at more than 40 textbooks on EM, mathematical physics, and potential theory, but not in one could I find the solution to the problem. A few books even went so far as to claim that the sum over k of the product of the functions in Eq. (2) (where the $k = 0$ term is excluded) is the *most general solution* to Laplace's equation! This is a common pitfall in which it is easy to fall, as I did in my EM course. It turns out that it is only in Cartesian coordinates that such a pitfall develops. In the spherical coordinate system all the solutions (including the $k = 0$ term) turn out to be Legendre polynomials. In the cylindrical coordinate system one ends up with the Bessel functions (including that of the zeroth order). However, it is the peculiarity of the Cartesian coordinate system that differentiates between the $k = 0$ term and the other terms in a substantial and nontrivial way. It is this differentiation which is not emphasized in the standard literature, and it is the purpose of this paper to emphasize it.

In what follows I intend to outline the solution. With zero as a possible value for k , the most general potential function can be written as

$$\Phi(x,y) = (A_0 x + B_0)(C_0 y + D_0) \\ + \sum_{k \neq 0} (A_k e^{kx} + B_k e^{-kx}) \\ \times (C_k \cos ky + D_k \sin ky),$$

where the new term is the product of the solutions to

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = 0.$$

We can now safely apply the BCs. The third BC in (5) gives

$$\Phi(x,0) = 0 = (A_0 x + B_0) D_0 \\ + \sum_{k \neq 0} (A_k e^{kx} + B_k e^{-kx}) C_k$$

leading to $D_0 = 0 = C_k$. Hence, the potential reduces to

$$\Phi(x,y) = (A_0 x + B_0) y + \sum_{k \neq 0} (A_k e^{kx} + B_k e^{-kx}) \sin ky,$$

where we have absorbed the multiplicative constants into the A 's and the B 's. Now apply the last BC in (5) to obtain

$$\Phi(x,b) = V_0 = (A_0 x + B_0) b \\ + \sum_{k \neq 0} (A_k e^{kx} + B_k e^{-kx}) \sin kb.$$

Since this equation is to hold for arbitrary x , we must have

$$A_0 = 0, \quad B_0 = V_0/b, \quad k = n\pi/b, \quad n = 1, 2, \dots$$

and the potential function becomes

$$\Phi(x,y) = \left(\frac{V_0}{b}\right) y + \sum_{n=1}^{\infty} (A_n e^{(n\pi/b)x} + B_n e^{-(n\pi/b)x}) \\ \times \sin[(n\pi/b)y]. \quad (6)$$

It is already clear from (6) that the linear term in y will be absent only if $V_0 = 0$, i.e., if Φ is periodic in y . Now we can apply the first BC to obtain

$$V_1 = \left(\frac{V_0}{b}\right) y + \sum_{n=1}^{\infty} (A_n + B_n) \sin\left(\frac{n\pi}{b} y\right).$$

Inverse Fourier transforming this leads to

$$A_n + B_n = \frac{2}{b} \int_0^b \left(V_1 - \frac{V_0}{b} y\right) \sin\left(\frac{n\pi}{b} y\right) dy \\ = (2/n\pi) [V_1 + (-1)^n (V_0 - V_1)]. \quad (7a)$$

Similarly, the second BC in (5) yields

$$A_n e^{n\pi a/b} + B_n e^{-n\pi a/b} \\ = \frac{2}{b} \int_0^b \left(V_2 - \frac{V_0}{b} y\right) \sin\left(\frac{n\pi}{b} y\right) dy \\ = (2/n\pi) [V_2 + (-1)^n (V_0 - V_2)]. \quad (7b)$$

We can solve these equations to obtain

$$A_n = \frac{1}{n\pi \sinh(n\pi a/b)} \{ [V_2 + (-1)^n (V_0 - V_2)] \\ - [V_1 + (-1)^n (V_0 - V_1)] e^{-n\pi a/b} \}, \\ B_n = \frac{1}{n\pi \sinh(n\pi a/b)} \{ [V_1 + (-1)^n (V_0 - V_1)] e^{n\pi a/b} \\ - [V_2 + (-1)^n (V_0 - V_2)] \}.$$

Substituting these in (6) and simplifying, we obtain

$$\Phi(x,y) = \frac{V_0}{b} y + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n \sinh(n\pi a/b)} \\ \times \{ v_n^{(2)} \sinh(n\pi x/b) \\ - v_n^{(1)} \sinh[(n\pi/b)(x-a)] \} \sin(n\pi y/b),$$

where

$$v_n^{(1)} \equiv [V_1 + (-1)^n (V_0 - V_1)]$$

and

$$v_n^{(2)} \equiv [V_2 + (-1)^n (V_0 - V_2)].$$

New wine into old bottles: A nuptial arch

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At a recent wedding that I attended, I observed an arch on which balloons were clustered like grapes on a vine. The arch itself spanned a very large section of a very large hall. The question that naturally arose was: What was the shape of this arch and what was holding it up? Close inspection showed that it was made entirely of helium balloons that were closely and symmetrically arranged around each segment of, and along the length of, a long thin string that was tied down at each end. A little reflection made it clear that this was a well-known problem cast in a new setting.

The old problem being referred to here is the catenary, i.e., the shape that a uniform cable takes when it supports its own weight.¹ Consider a uniform heavy cable of weight w per unit length suspended from two points at the same height. Let T_0 be the tension at the lowest point of the cable. If a Cartesian set of axes is drawn so that the y axis coincides with the axis of symmetry of the cable, then the equation representing the shape of the cable is

$$y = c \cosh(x/c) + b. \quad (1)$$

Here, $c = T_0/w$ and b is a constant. It is customary to set b equal to 0 in which case the x axis, now at a distance c below the lowest point of the cable, is the directrix of the catenary. Note that x is restricted to $|x| \leq a/2$, where a is the straight line distance between the two end points.

In the new setting for this problem, each of the balloons was subject to a buoyant force that was greater than its weight. The net effect was equivalent to having a light uniform string subject to a rather large upward gravitational field. This in turn gave rise to the nuptial catenary.

We can represent the above by choosing the x axis to

coincide with the horizontal line between the end points, which are a distance a apart. Then Eq. (1) can be replaced by

$$y = c [\cosh(a/2c) - \cosh(x/c)]. \quad (2)$$

Again $|x| \leq a/2$ and $c = T_0/w$ except now T_0 is the tension at the highest point of the string, while w refers to the buoyant force per unit length of string minus the actual weight of balloons and string per unit length of string. The actual value of c can be determined readily from the length l of the catenary and the central height h . In particular, since $l = 2c \sinh(a/2c)$ and $h = c [\cosh(a/2c) - 1]$, it follows directly from the identity $\cosh^2(a/2c) - \sinh^2(a/2c) = 1$ that $c = (l^2 - 4h^2)/8h$.

An interesting similar example of the above can also be found in a stone bridge.² The arch of such a bridge takes the shape of a catenary if the stress at any point of the arch, due to the weight it supports, is tangent to the arch at this point.

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¹R. A. Becker, *Introduction to Theoretical Mechanics* (McGraw-Hill, New York, 1954), p. 75.

²L. M. Kells, *Elementary Differential Equations* (McGraw-Hill, New York, 1954), p. 139.

Comment on "Linear least-squares fits with errors in both coordinates," by B. C. Reed [Am. J. Phys. 57, 642-646 (1989)]

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In a recent paper, Reed¹ clarified several misleading statements in York's article² on finding the least-squares straight line to fit data with errors in both coordinates and studied the method of finding the proper root of the least-

squares cubic. There have been many papers treating the generalized least-squares problem for data with errors in both coordinates,³ including some that consider y as a polynomial in x or even as a function of x which is nonlinear in