On Weak Ground

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Abstract

Though the study of grounding is still in the early stages, Kit Fine, in "The Pure Logic of Ground", has made a seminal attempt at formalization. Formalization of this sort is supposed to bring clarity and precision to our theorizing, as it has to the study of other metaphysically important phenomena, like modality and vagueness. Unfortunately, as I will argue, Fine ties the formal treatment of grounding to the obscure notion of a *weak ground*. The obscurity of weak ground, together with its centrality in Fine's system, threatens to undermine the extent to which this formalization offers clarity and precision. In this paper, I show how to overcome this problem. I describe a system, the logic of strict ground (LSG) and demonstrate its adequacy; I specify a translation scheme for interpreting Fine's weak grounding claims; I show that the interpretation verifies all of the principles of Fine's system; and I show that derivability in Fine's system can be exactly characterized in terms of derivability in LSG. I conclude that Fine's system is reducible to LSG.

A number of contemporary metaphysicians are exploring the nature and uses of a phenomenon known as *grounding*. These thinkers link grounding to a certain kind of explanation. Philosophers and scientists are fond of asking for explanations of this kind: "In virtue of what is murder wrong?" "In virtue of what am I justified in believing that I have hands?" "What makes gravity such a weak force?" Each question sets the stage for a more or less familiar ongoing research program. Each question calls for an explanation. A correct answer to each question will tell us which are the facts in virtue of which something is the case. In general, the facts that ground a given fact are the facts in virtue of which that fact obtains.[1]

[1] Enthusiasts of grounding agree that facts may ground other facts, but they split on

Proponents of grounding have claimed that it is the key to understanding certain traditionally recognized metaphysical investigations. Grounding is useful for helping us characterize and pursue investigations concerning realism and anti-realism in various domains [Fine, 2001]. Grounding provides a way of making sense of interesting kinds of metaphysical dependence [Correia, 2008]. It provides a way of articulating a sensible form of nominalism about properties [Melia, 2005]. It is the notion needed to characterize physicalism [Schaffer, 2009, 2003]. It provides a way of reconciling a sparse inventory of fundamental entities with the rich ontological commitments of the special sciences [Armstrong, 1997], [Cameron, 2008], [Schaffer, 2007, 2009, 2010]. In short a, or perhaps the, central concern of metaphysics is saying what grounds what, thereby limning the structure of reality.

Any tool with such disparate and important uses deserves close scrutiny. An investigation into the susceptibility of grounding to formal treatment is an important component of such scrutiny. Though the study of grounding is still in the early stages, Kit Fine, in "The Pure Logic of Ground" [Fine, 2012a], has made a seminal attempt at formalization. Formalization of this sort is supposed to bring clarity and precision to our theorizing, as it has to the study of other metaphysically important phenomena, like modality and vagueness. Unfortunately, as I will argue, Fine ties the formal treatment of grounding to the obscure notion of a *weak ground*. The obscurity of weak ground, together with its centrality in Fine's system, threatens to undermine the extent to which this formalization offers clarity and precision. In this paper, I show how to overcome this problem.

whether grounding relates things other than facts; Schaffer [2009] contends that it does, and Fine [2001, 2012a] disagrees. If Schaffer is correct, then Fine's formal treatment of ground may be incomplete. I take no stand on this dispute here. Fine also argues in [Fine, 2001] that, strictly speaking, we don't need to reify facts and claim that grounding is a relation between them in order to give a theory of ground; we may instead treat talk of grounding's being a relation between facts as a mere *façon de parler*. We should formalize our theory of ground by appeal to sentential operators which do not pick out any relation, and whose arguments, semantically speaking, do not pick out entities. According to Fine, the logical treatment of such a sentential operator need not commit us to an ontology of facts. It should be noted, however, that Fine's semantic treatment in [Fine, 2012a] appeals to an ontology of facts. I will follow Fine in couching what follows in terms of facts and the grounding relations they may bear to one another. I presume that, if Fine's semantics can be understood so as not to engage the ontology of facts, what I say can similarly be recast.

1 Weak Ground in the Pure Logic of Ground

In Fine's system, which he calls *the pure logic of ground* (PLG), grounding claims vary along two independent axes: (*i*) a grounding claim may either be *partial* or *full*; and (*ii*) it may either be *strict* or *weak.* There are thus four types of grounding claim in Fine's system. Corresponding to each type of grounding claim is a symbol in the language of PLG:

	strict	**weak**
full	$<$	$\leq$
partial	$\prec$	$\preceq$

The difference between a *full* grounding claim and a corresponding *partial* grounding claim is relatively clear: a partial ground for a fact ϕ is some fact that is a (perhaps improper) part of a full ground for ϕ: a partial ground of ϕ is either itself a full ground for ϕ, or is one of a plurality of facts that fully ground ϕ.[2] And the notion of a strict ground for ϕ is the relatively familiar notion of a fact *in virtue of which* ϕ obtains.

The notion of weak ground is obscure by comparison. What is a weak ground for a fact ϕ? Fine takes two claims about weak ground as axiomatic: (*i*) that every strict ground for ϕ is also a weak ground for ϕ; and (*ii*) that every fact is a weak ground for itself. This second claim marks a contrast with strict ground, which is taken by Fine (along with most other philosophers working on ground) to be irreflexive. Still, these two claims fall far short of a characterization of what weak ground comes to.

The obscurity, or at least unfamiliarity, of the notion of weak ground is discomfiting in large measure because of the importance the notion has in developing PLG. According to Fine, "it turns out that the most natural way of developing a logic of strict ground is by combining it with the logic of weak ground" (p. 1). If PLG represents the best way to develop a logic of ground, and PLG crucially relies on a notion that is unclear or problematic, then we might worry that the logic of ground is itself problematic. It is difficult to assess

[2]This rough and ready explanation of the notion of a partial grounding claim explains that notion in terms of full grounding. I do not mean by these informal remarks to undertake any commitment on the interesting question of whether partial grounding is ultimately to be analyzed in terms of full grounding. The characterization in the main text is deployed merely as an intuitive aid to the reader's understanding. Fine specifies a number of distinct notions of partial ground, not all of which can be characterized in the way suggested in the main text. See [Fine, 2012b, pp. 53-4] in particular for an argument that strict partial ground cannot be characterized in that way.

the plausibility of the logical principles on offer if we don't understand one of the notions used to frame those principles. Worse, if the best way of giving a formal treatment of the notion of strict ground turns out to rely on an unclear notion, then we might worry that the notion of strict ground also stands in need of clarification. The worry here is akin to Quine's complaint that modal logic yielded only an "illusion of understanding" of the notion of necessity, since, he claimed, it relied on a confusion of use and mention [Quine, 1966, p. 176]. If formal treatment of strict ground relies on an obscure notion, then we have reason to suspect that the notion so treated is obscure.

Fine offers two characterizations of weak ground. There are two further characterizations that may be gleaned from what he says. Still, as I will argue in the next section, none of these characterizations offers a satisfactory explication of the notion of weak ground.

Fine never claims that weak ground is indispensable for formulating the logic of ground. Instead he makes the weaker claim that treating the logic of strict ground in isolation leads to "anomalies in the formulation", and that combining the treatment of weak ground with a treatment of strict ground turns out to provide a more natural starting point for the logic of ground [Fine, 2012a, p. 1]. This leaves open the hope of offering a formulation of the logic of ground that relies only on the notion of strict ground. In this paper, I show how this hope may be fulfilled, outlining a way of treating the logic of strict ground in isolation, and interpreting weak grounding claims in terms of strict grounding claims so that the axioms and inference rules of PLG are verified. I describe a fragment of PLG, which I call *the logic of strict ground* (LSG) and a translation scheme for representing the grounding claims in the language of PLG in the language of LSG; I show that the scheme verifies all of the principles of PLG on the proposed interpretation; and I show that the derivability of a grounding claim from a set of premises in PLG can be exactly characterized in terms of the derivability of its interpretations in LSG. I conclude that PLG is reducible to LSG.

This is good news for theorists of ground. LSG is simpler than PLG in at least two ways. First, its language only contains two grounding operators, one for full strict ground and the other for partial strict ground. So, it relies on fewer primitives. Second, it relies on fewer inference rules, and each of the rules of LSG is also valid in PLG. Thus, the formulation of the logic of ground afforded by LSG is at least as elegant as the formulation in PLG. What's more, because LSG contains only expressions for strict ground, it does not rely on a primitive notion of weak ground. If, as I will argue, the notion of weak ground stands in need of clarification, then LSG will be clearer on this score than PLG. The results derived here, however, show that there is *a* notion that is analyzable in terms of strict ground, and which satisfies the inference rules of PLG. So, we can *introduce* a notion of weak ground, in effect defined in terms of strict ground, and show that the notion, so-defined, yields the pure logic of ground. If this notion captures everything of importance in the notion of weak ground deployed by Fine, then we will have thereby provided a clarification of that notion. So, the results we will prove show that LSG is simpler than and at least as clear and plausible as PLG.

2 What is Weak Ground?

Fine offers two intuitive characterizations of weak ground. Fine's semantics suggests another characterization. A remark suggests yet another. There is reason to think that none of these four characterizations provides an adequate explanation of the notion of weak ground.[3]

Fine first suggests that, while a strict ground for ϕ is a fact occuring lower than ϕ in the explanatory hierarchy, a weak ground for ϕ is a fact occurring *at the same level as* ϕ in the explanatory hierarchy. Theorists of ground have defended the idea that grounding makes available a creditable explication of the idea of an explanatory hierarchy organized into levels.[4] But it is not clear that being at the same level as ϕ in an explanatory hierarchy of this sort corresponds to any useful notion of ground. Suppose it's chilly, but neither windy nor sunny. Then, it is either chilly or windy in virtue of the fact that it is chilly; similarly, it is either chilly or sunny in virtue of the fact that it is chilly. Its being either

[3]My review of the first three characterizations pushes them farther than I believe Fine intends. On my reading, Fine does not intend any of these three characterizations as a full explication of the notion of weak ground, so much as an interesting further fact concerning the relation picked out by the notion that may help us understand the idea. The fourth characterization is intended to be a definition of weak ground in terms of strict ground. Nevertheless, the question of whether any of these characterizations can be used to specify the notion is worth exploring. The arguments that follow, if sound, provide reasons not only for rejecting the full explications of the notion of weak ground suggested by the characterizations, but also for finding the characterizations less helpful for understanding weak ground than one might have hoped.

[4]See, *e.g.*, [deRosset, 2013], [Schaffer, 2007, 2010]

chilly or windy occurs at the same level in the explanatory hierarchy as its being either chilly or sunny. But, one may reasonably feel, there is no explanatory relation between them that one would want to classify as a kind of ground, akin to the *in virtue of* relation.

Fine's second intuitive characterization of the notion of weak ground is in terms of the English idiom "For ..., for ...". He writes, "[t]hus for John to marry Mary is for John to marry Mary, for John to marry Mary is for Mary to marry John, and for John to marry Mary is for John to marry Mary and (for) Mary to marry John. In each of these cases, we may say that the truth or truths on the right weakly ground the truth on the left." [Fine, 2012a, p. 3] This characterization is unsatisfactory, given that strict grounds are also supposed to be weak grounds. To my mind, it would be clearly false to say that for it to be either chilly or windy is for it to be chilly, in part because the former fact may obtain when the latter does not.

A quick glance at the semantics Fine offers for PLG gives us a third characterization of weak ground. His semantics relies on an ontology of *facts*, which are to be thought of as "...*parts* of the actual world" [Fine, 2012a, p. 7, emphasis original]. The sentences of the language are all truths, and each such sentence is to be associated with a *verification set* – the set of facts that make it true (or "truthmakers," as such things are called in the literature).[5] On the conception of weak ground sugggested by Fine's semantics, the full weak grounding claim $\phi \leq \psi$ is true just in case every member of ϕ's verification set is also a member of ψ's, *i.e.*, every truthmaker for ϕ is also a truthmaker for ψ.[6] Suppose again that it is chilly but not sunny. According to Fine's semantics, its being chilly is a weak ground for its being either chilly or sunny just in case every truthmaker for 'it's chilly' is also a truthmaker for 'it's either chilly or sunny.' So far, so good. A problem arises, however, if we allow that the truthmakers may be *sparse* in such a way that the only parts of the actual world which are truthmakers for a true disjunction with a false disjunct are the truthmakers for its true disjunct. Such a view is intuitively attractive, and many in the truthmaker

[5]There are truthmaker theorists who deny that truthmakers are facts; see [Mulligan et al., 1984] for a classic example. The argument of this paragraph could easily be amended to accommodate alternative views concerning the nature of truthmakers.

[6]This is a special case of the more general truth condition given on [Fine, 2012a, p. 9], in which the left-hand side of the weak grounding claim is a single sentence ϕ; Fine's truth condition is stated in terms of sequents with an arbitrary set of sentences $\phi_0, \phi_1, ...$ on the left-hand side.

literature endorse it. But, if the truthmakers are sparse, then Fine's semantics will imply that its being either chilly or sunny is a weak ground for its being chilly. As above, one may reasonably feel that there is no explanatory relation going from the disjunction to its disjunct that one would want to classify as a kind of ground, akin to the *in virtue of* relation. In fact, it seems very clear that the explanatory relation between the true disjunct and the disjunction is asymmetric.[7] It would appear, then, that the adequacy of this particular explanation of weak ground rules out a view of truthmakers that ought not to be ruled out.[8]

Fine makes one more claim concerning weak ground, that one might hope would helpfully specify the notion. Fine writes, "[i]n general, whenever a number of truths [fully] weakly ground a given truth, whatever explanatory role can be played by the given truth can also be played by their grounds" (p. 3).[9] This claim describes the explanatory role the weak grounds for a fact may play. Fine suggests that we may use this specification of the explanatory role that full weak grounds play to define the notion of weak ground [Fine, 2012b, p. 52].

The idea is that some facts weakly ground ϕ if and only if they strictly ground (perhaps in concert with some further facts Γ) all of the facts strictly grounded (perhaps in concert with Γ) by ϕ.[10] Assuming (as PLG requires) that

[7]This objection presupposes that the facts available to serve as truthmakers are restricted to those that actually obtain. Although Fine endorses the presupposition [Fine, 2012a, p. 7], nothing in the formal apparatus of his semantics requires it. We might hope, then, to avoid the problem by "going possibilist," *i.e.*, appealing to facts that either actually obtain or might have. Because it might have been sunny today, there is a possible fact which may serve as an additional truthmaker for 'it's either chilly or sunny.' We will run into similar problems, however, handling cases of true disjunctions with impossible disjuncts, like 'either it is chilly or 0=1.' It would seem, then, that Fine's semantics will ultimately avoid the problem only if we allow ourselves to "go impossibilist" by postulating *impossible* facts to serve as truthmakers for true disjunctions. Fine has, in another context, given a truthmaker semantics that appeals to an infinite number of impossible facts, or, as he calls them, "contradictory states" [Fine, 2014, §4]. Thanks to an anonymous referee.

[8]In personal correspondence, Fine confirms that the semantics for PLG presupposes a plenitudinous ontology of facts, on which there is a one-one relation between true sentences and the facts they report.

[9]I have interpolated "fully" into this characterization in order to deal with certain uninteresting counterexamples to the unamended claim. Suppose, for instance, that P merely partially weakly grounds Q. Q fully strictly grounds $(Q \vee R)$, but we wouldn't expect it to follow that P by itself fully strictly grounds $(Q \vee R)$; given that P, Q, and $(Q \vee R)$ are sentences, in PLG $\{P \preceq Q, Q < (Q \vee R)\}$ does not imply $P < (Q \vee R)$. My interpolation does no damage to the utility of the characterization since, if we had an adequate characterization of full weak ground, we could explain the notion of partial weak ground in its terms in the way suggested at the beginning of this paper: a partial weak ground of ϕ is some fact that is a (perhaps improper) part of a full weak ground for ϕ .

[10]Formally, the definition can be captured in model-theoretic terms. Let f and g be verification sets and F and G be sets of verification sets of a model $\mathcal{M}$. Then the statement of the

grounding is transitive, every strict ground of ϕ satisfies this characterization. But there is at least one fact that also satisfies this characterization and which isn't a strict ground of ϕ: ϕ itself. So, this specification satisfies the claims concerning weak ground that Fine takes as axiomatic.

Problems arise, however, when we confront situations in which the universal generalization used to define the notion of weak ground is vacuously satisfied. Suppose, for instance, that there are only two "atomic" facts a and b, and just one "conjunctive" fact $a.b$. The conjunctive fact $a.b$ does not strictly ground any facts, since, on pain of circularity, it strictly grounds neither itself nor its conjuncts, and these exhaust the facts. Thus, according to the definition, a is a (full) weak ground of $a.b$. But, one may reasonably feel, there is no full explanation of the conjunction that appeals to only one conjunct and that one would want to classify as a kind of ground, akin to the *in virtue of* relation.

It is worth noting in this connection that Fine's semantics appears to be at odds with the proposed definition of weak ground in this case. On that semantics, there is a model in which, roughly, there are only the three facts a, b, and $a.b$.[11] On Fine's semantics, it is not true in this model that a is a full weak ground of $a.b$, even though everything strictly grounded by $a.b$ – namely, nothing – is also strictly grounded by a. I've argued that the semantics is correct on this point. If I'm wrong about that, though, the problem of the mismatch between the proposed definition and its semantic implementation remains.

The problem cannot be fixed by requiring *non-vacuous* satisfaction of the explanatory role played by $a.b$, since then $a.b$ would no longer weakly ground itself. There are a variety of more complicated maneuvers that might avoid the problem. For instance, we could stipulate as a background condition on the definition that chains of strict grounds have no "top", so that every fact strictly grounds some further fact. (However, this would have the effect of

definition will be:

$$G \leq_{\mathcal{M}} f \text{ iff } (f, F <_{\mathcal{M}} g \Rightarrow G, F <_{\mathcal{M}} g).$$

See [Fine, 2012a, p. 9] for a specification of the relation between the notions expressed by $<_{\mathcal{M}}$ and $\leq_{\mathcal{M}}$ and the object-language grounding symbols $<$ and $\leq$.

[11] Less roughly, there is a fact frame in which the universe of facts is $\{n, a, b, a.b\}$, and the fusion operation Π is such that $\Pi(\emptyset) = n$, $\Pi(\{a, b\}) = a.b$, $\Pi(\{x\}) = x$, $\Pi(\{n\} \cup \Gamma) = \Pi(\Gamma)$, and $\Pi(\{a.b\} \cup \Gamma) = a.b$. (Fine's semantics requires the existence of a "null fact," and the element n is playing that role here.) Suppose our language has only two sentences ϕ and ψ, respectively. Then a model $\mathcal{M}$ for that language relative to our fact frame is given by letting the interpetation $[\phi]$ of ϕ be $\{a\}$ and the interpretation $[\psi]$ of ψ be $\{a.b\}$. In this model $\mathcal{M}$, $[\phi] \not\leq_{\mathcal{M}} [\psi]$, and so $\mathcal{M} \nvDash \phi \leq \psi$.

ruling out the hypothesis that there are only finitely many facts.)[12] I won't pause to chase down the myriad ways in which one might respond to the problem for the proposed definition of weak ground. The important point for present purposes is that a response is needed. As it stands, Fine's proposed definition of weak ground encounters problems, and does not appear to match its semantic implementation.

I have offered only a very short review of some initial difficulties with each of the four characterizations of weak ground. None of these difficulties are decisive. What's more, there are various other proposals for defining weak ground in terms of strict ground which I have not discussed here. Indeed, in what follows I propose, in effect, one such definition. The point for present purposes is that our attempts to characterize weak ground have led us into difficulties. We can avoid these difficulties altogether if we can provide a formulation of the logic of strict ground which does not rely on the notion of weak ground. This is what I propose to do next. I start by laying out the two systems we will be studying.

3 The *Pure Logic of Ground* (PLG)

We turn first to a review of Fine's pure logic of ground (PLG). A *language* $\mathcal{L}$ of PLG is a tuple $\langle S, <, \leq, \prec, \preceq \rangle$, such that S is non-empty set, and $<, \leq, \prec$, and $\preceq$ are symbols indicating the four kinds of ground introduced in §1. The members of S are the *sentences* of $\mathcal{L}$. Intuitively, we may think of the sentences in S as expressions that state facts. PLG is a sequent calculus, in which these sentences are combined with the grounding symbols to construct more complicated syntactic objects, called *sequents*. In PLG these sequents represent grounding relations among facts. To ensure that sequents of $\mathcal{L}$ have a unique parsing, we stipulate that the grounding symbols are pairwise distinct and not in S. Fine defines the sequents of $\mathcal{L}$ as follows:

Definition For all $\Delta \subseteq S$ and $\phi, \psi \in S$, the following are exactly the *sequents* of $\mathcal{L}$:

$$\Delta < \phi \qquad \Delta \leq \phi \qquad \psi \prec \phi \qquad \psi \preceq \phi$$

Much of our discussion will turn on the structure of sets on the left-hand sides of sequents whose operator is either $\leq$ (full weak grounding) or $<$ (full strict

[12]The claim that chains of strict grounds have no "top" is more plausible if we "go possibilist" in the sense of n.7.

grounding). Following Fine (and one tradition in the presentation of sequent calculi), in a specification of the form of such a sequent, a comma on the left indicates set-theoretic union, and a sentence variable on the left indicates the relevant sentence's singleton. For instance, if Δ and Γ are sets of sentences, and ϕ and ψ are sentences, then

$$\Delta, \Gamma \leq \psi \qquad \phi < \psi \qquad \Delta, \phi \leq \psi$$

indicate the sequents

$$\Delta \cup \Gamma \leq \psi \qquad \{\phi\} < \psi \qquad \Delta \cup \{\phi\} \leq \psi$$

respectively.

The *inference rules* of PLG are given in figure 1. The '$\perp$' in the statement of non-circularity is not a distinguished sequent standing for falsity, but rather indicates that any sequent whatsoever is derivable from a sequent of the form $\phi \prec \phi$.[13] A *derivation* in PLG of a sequent σ from a set of sequents S is a converse well-founded tree[14] whose root is σ, whose leaves are either members of S or instances of IDENTITY, and whose nodes follow from their predecessors by one of the inference rules of PLG. I will write $S \vdash \sigma$ to mean that there is a derivation in PLG of σ from S.

4 The *Logic of Strict Ground* (LSG)

Given a language $\mathcal{L}$ of the pure logic of ground, a corresponding language $\mathcal{L}^s$ for the logic of strict ground can be obtained. The set of *sequents* of $\mathcal{L}^s$ is the set of sequents of $\mathcal{L}$ whose symbol for grounding is $<$ or $\prec$.

The inference rules for LSG are given in figure 2. All of the inference rules of LSG are derivable in PLG. The notion of a *derivation* in LSG is defined similarly to the notion of a derivation in PLG, except that nodes in a derivation follow by the axioms of LSG, rather than PLG. I will write $S \vdash^s \sigma$ to mean that there is a derivation in LSG of σ from S.

[13] Here and throughout I indulge in sloppiness about distinguishing use and mention. Adding quotation marks and corner quotes in every relevant place would complicate the presentation without aiding understanding.

[14] A converse well-founded tree is a tree for which the converse of the "child of" relation on the tree's nodes is well-founded. That is, there are no infinite chains starting at the root and extending up along any branch. Thus, every branch terminates in finitely many steps.

SUBSUMPTION

$$(</\leq)\text{: } \frac{\Delta < \phi}{\Delta \leq \phi} \qquad (\prec/\preceq)\text{: } \frac{\phi \prec \psi}{\phi \preceq \psi}$$

$$(</\prec)\text{: } \frac{\Delta, \phi < \psi}{\phi \prec \psi} \qquad (\leq/\preceq)\text{: } \frac{\Delta, \phi \leq \psi}{\phi \preceq \psi}$$

TRANSITIVITY

$$(\preceq/\preceq)\text{: } \frac{\phi \preceq \psi \qquad \psi \preceq \theta}{\phi \preceq \theta} \qquad (\preceq/\prec)\text{: } \frac{\phi \preceq \psi \qquad \psi \prec \theta}{\phi \prec \theta}$$

$$(\prec/\preceq)\text{: } \frac{\phi \prec \psi \qquad \psi \preceq \theta}{\phi \prec \theta}$$

IDENTITY

$$\overline{\phi \leq \phi}$$

NON-CIRCULARITY

$$\frac{\phi \prec \phi}{\bot}$$

CUT

$$\frac{\Delta_0 \leq \phi_0 \qquad \Delta_1 \leq \phi_1 \qquad \ldots \qquad \phi_0, \phi_1, \ldots \leq \phi}{\Delta_0, \Delta_1, \ldots \leq \phi}$$

REVERSE SUBSUMPTION

$$\frac{\phi_0, \phi_1, \ldots, \leq \phi \qquad \phi_0 \prec \phi \qquad \phi_1 \prec \phi \qquad \ldots}{\phi_0, \phi_1, \ldots < \phi}$$

Figure 1: The pure logic of ground

For the remainder of this paper, I will assume we are given a language $\mathcal{L}$, and I will suppress reference to $\mathcal{L}$, and just talk about sentences and sequents.

It is obvious that derivations that do not use NON-CIRCULARITY in either PLG or LSG preserve syntactic information in ways that derivations that use NON-CIRCULARITY may not. For instance it is straightforward to show that any sentence occurring in any sequent σ derivable from S without NON-CIRCULARITY (in either PLG or LSG) also occurs in some sequent in S, so long as σ is not a *weak identity* (*i.e.*, not of the form $\phi \leq \phi$ or $\phi \preceq \phi$). Thus, many of the results below concern derivations that contain no application of NON-CIRCULARITY. I will call such derivations (either in PLG or LSG) *normal derivations*; I will use '$\vdash^{-}$' to indicate the existence of a normal derivation in PLG; and '$\vdash^{s-}$' to indicate the existence of a normal derivation in LSG. I will

SUBSUMPTION
$$\frac{\Delta, \phi < \psi}{\phi \prec \psi}$$

TRANSITIVITY
$$\frac{\phi \prec \psi \qquad \psi \prec \theta}{\phi \prec \theta}$$

NON-CIRCULARITY
$$\frac{\phi \prec \phi}{\bot}$$

CUT
$$\frac{\Delta_0 < \phi_0 \qquad \Delta_1 < \phi_1 \qquad \ldots \qquad \Gamma, \phi_0, \phi_1, \ldots < \phi}{\Gamma, \Delta_0, \Delta_1, \ldots < \phi}$$

AMALGAMATION
$$\frac{\Delta_0 < \phi \qquad \Delta_1 < \phi \qquad \ldots}{\Delta_0, \Delta_1, \ldots < \phi}$$

Figure 2: The logic of strict ground

follow Fine in calling a set of sequents S *consistent* (in either PLG or LSG) if no sequent of the form $\phi \prec \phi$ is derivable from S. Fine shows that S is inconsistent in PLG iff there is a normal derivation of a sequent of the form $\phi \prec \phi$ from S [Fine, 2012a, p. 11, Lemma 4.2]. Though Fine does not explicitly consider LSG, his proof extends straightforwardly to that system.[15]

5 Adequacy of LSG for the Pure Logic of Ground

Fine offers a semantics for PLG, and shows that PLG is sound and complete for that semantics. We can use Fine's construction and results to offer a semantics for LSG and prove corresponding soundness and completeness results. The task is made easy by the fact that all of the rules of LSG are also rules of PLG, and that all sequents of LSG are also sequents of PLG. Fine defines the notions of a *generalized fact model* of PLG and a *fact model* of PLG [Fine, 2012a, p. 8]. I will say that $\mathcal{M}$ is a *model* of PLG just in case it is either a generalized fact

[15]This result implies that normal derivability may play a special role in our account of derivability in PLG. For it implies that derivability in PLG is reducible to normal derivability in PLG: a sequent is derivable from S iff either it is normally derivable from S or there is a sequent of the form $\phi \prec \phi$ that is normally derivable from S.

model or a fact model of PLG. Fine defines the notion of a sequent σ of PLG's being *true in* a model $\mathcal{M}$ ($\mathcal{M} \models_{PLG} \sigma$). The notion of σ's being a consequence of S ($S \models_{PLG} \sigma$) is then defined in the standard way as truth-preservation in every model. Given these notions, it is easy to define correlative semantic notions for LSG:

- $\mathcal{M}$ is a *model* of LSG iff $\mathcal{M}$ is a model of PLG.
- A sequent σ of LSG is *true in* a model $\mathcal{M}$ ($\mathcal{M} \models^s \sigma$) iff $\mathcal{M} \models_{PLG} \sigma$;
- a set S of sequents is *true in* a model $\mathcal{M}$ ($\mathcal{M} \models^s S$) iff $(\forall \sigma \in S)\mathcal{M} \models^s \sigma$; and
- a sequent σ is a *consequence of* a set of sequents S ($S \models^s \sigma$) iff for all models $\mathcal{M}$, if $\mathcal{M} \models^s S$ then $\mathcal{M} \models^s \sigma$.

This gives us a semantics for LSG. It is then straightforward to prove

Theorem C.1 ***(Soundness)*** *If $S \vdash^s \sigma$, then $S \models^s \sigma$*

and

Theorem C.4 ***(Completeness)*** *If $S \models^s \sigma$, then $S \vdash^s \sigma$.*

Because LSG is sound and complete for Fine's semantics, LSG suffices to exactly characterize those logical relations of consequence among strict grounding claims captured by PLG. The salutary upshot is that neither the language nor the axiomatization of the logic of strict ground need appeal to Fine's notion of weak ground. Moreover, the proof of theorem C.4 proceeds in effect by showing that PLG is a conservative extension of LSG.[16] Given the consistency of LSG, this guarantees that there is some interpretation of weak grounding claims that validates the axioms of PLG. So, if we want to continue to use PLG and the notion of weak ground its axioms characterize, we may do so in good conscience.[17]

It would be nice, however, to be able to helpfully specify an interpretation of weak grounding claims that validates the axioms. Fine's remarks suggested four interpretations, but each of those interpretations faces problems. I now turn to the task of providing a helpful interpretation of weak grounding claims that avoids the problems.

[16] Thanks to an anonymous referee for highlighting the need to emphasize this point.
[17] See [Fine, 2012b, pp. 63ff.] for a proposed application.

6 An Interpretation of Weak Ground

I aim to offer a translation scheme for weak sequents in PLG into sequents in LSG, so that (*i*) the inference rules of PLG are validated, in the sense that, if a sequent is derivable from some premises in PLG, then a translation of that sequent is derivable from a translation of those premises in LSG; and (*ii*) the derivability of a sequent in PLG can be exactly characterized in terms of the derivability in LSG of its translation. We will interpret weak identity sequents as asserting the self-identity of ϕ; since such claims are trivial, we will ignore weak identities in our formal translation scheme. Here's the rough idea of the translation scheme for other sequents of PLG: we translate strict sequents homophonically; we translate a partial weak sequent $\phi \preceq \psi$ by the corresponding partial strict sequent $\phi \prec \psi$; we translate a full weak sequent $\Delta \leq \psi$ as the corresponding full strict sequent $\Delta < \psi$ if Δ does not contain ψ, and as the claim that all of the members of Δ other than ψ are jointly part of a strict full ground for ψ otherwise.

The case of a full weak sequent of the form $\Delta, \psi \leq \psi$ ($\Delta \neq \emptyset, \psi \notin \Delta$) is obviously the most complicated and bears comment. Let's take as our example a sequent $\phi, \chi, \psi \leq \psi$ (where ϕ, χ, and ψ are all distinct from one another). Our translation takes this sequent to express the claim that ϕ and χ are jointly part of a strict full ground of ψ. That is, there is some other, unspecified fact that, together with ϕ and χ, fully strictly ground ψ. This translation is more complicated than might seem necessary. Why not take the ordinary arithmetical meaning of '$\leq$' as our guide, and interpret a weak full grounding claim as saying that the facts on the left-hand side *other than* ψ strictly ground ψ? In the case at hand, the proposal would be that $\phi, \chi, \psi \leq \psi$ expresses the claim that ϕ and χ by themselves fully ground ψ.

This translation scheme is simpler, but it's wrong. Suppose, for instance, that it is chilly, windy, and sunny, and consider the claim that its being chilly, windy, and sunny is weakly grounded in the triple of facts: it is chilly; it is windy; it is chilly, windy, and sunny.[18] It is clear that we should not interpret this claim as the claim that its being chilly, windy, and sunny is strictly and fully grounded in the pair of facts: it's chilly, it's windy. The problem isn't the

[18]Is this claim true? Well, for the reasons stated in §2, I am unsure whether this claim fits the notion of weak ground that Fine has in mind. According to the translation scheme I offer in this paper, it will turn out to be true.

implausibility of the translation, for we've got no reason as yet to think that the original weak grounding claim is plausible. The problem, rather, is that this translation does not adequately capture the logic (in PLG) of the original weak grounding claim. On the simple translation scheme on offer, $\phi, \chi, \psi \leq \psi$ and $\phi, \chi \leq \psi$ would get *the same interpretation*; thus, according to this interpretation, the latter should be trivially derivable from the former. But, in PLG $\phi, \chi, \psi \leq \psi$ does not imply $\phi, \chi \leq \psi$. We are left with a mystery concerning why PLG does not sanction the interderivability of these two claims, given that that they are synonymous according to the interpretation. In PLG, the removal of ψ from the left-hand side of the sequent might leave a "hole" that needs to be filled by some other sentence.[19] For this very reason, I doubt that there is a way to exhaustively characterize derivability in PLG in terms of derivability in LSG using this simple translation scheme. To vindicate the hope of reducing PLG to LSG, we need a more sophisticated translation, like the one I have suggested. But how are we to carry out the idea that $\phi, \chi, \psi \leq \psi$ expresses the claim that ϕ and χ are jointly part of a strict full ground of ψ?

The simplest expedient is to import, for each sentence ψ, a fresh sentence b_ψ, that represents a fact that "plugs any holes" that might be created when ψ is removed from the left-hand side of sequents of the form $\Delta, \psi \leq \psi$. Thus, we'll use b_ψ to interpret $\phi, \chi, \psi \leq \psi$ as $\phi, \chi, b_\psi < \psi$. We'll call these "hole-pluggers" *witnessing constants.*[20]

We will use witnessing constants to carry out the rough idea sketched above. First, let's define a notion that relates sets containing witnessing constants to those that don't.

[19]Here, perhaps, an analogy with mereological fusion may help. It is obvious that there will be cases in which an individual is a mereological fusion of itself and some other individuals, but the individual is not the mereological fusion of the other individuals. For instance, Michelangelo's *David* is the mereological fusion of *David*, *David's* head, and *David's* left leg; but *David* is not the mereological fusion of *David's* head, and *David's* left leg. Removing *David* from the list of fusees leaves a "hole" that needs to be filled by something else.

[20]This is the simplest expedient, but it is not the most natural. We are, in effect, assuming that a *single* "hole-plugging" fact will plug the holes left by the removal of ϕ from the right-hand sides of both $\psi, \phi \leq \phi$ and $\chi, \phi \leq \phi$. There is no reason to believe that there is such a single, all-purpose "hole-plugger" for ϕ. A more natural course would be to introduce a witnessing constant for each weak sequent in S of the form $\Delta, \phi \leq \phi$; thus we would have different "hole-plugging" witnessing constants for $\psi, \phi \leq \phi$ and $\chi, \phi \leq \phi$ if $\psi \neq \chi$. It is obvious, however, that the results we show below for our simpler translation scheme could also be shown, with a little added complication, for the more complicated but more natural translation scheme. Since the construction is already complicated enough, I am adopting the simpler expedient. The technique of witnessing constants is used in Fine's completeness proof for PLG [Fine, 2012a, pp. 14-6, 17-9].

Definition A set B_Δ is a *b-version* of a set Δ iff Δ is the result of uniform replacement of every witnessing constant $b_\phi \in B_\Delta$ by its corresponding sentence ϕ.

Now, let's use this notion to define the notion of a translation of a sequent σ. Suppose that σ is not a weak identity, $\phi \notin \Delta$, and B_Δ is a b-version of Δ. Let $\Delta \pm \phi$ ($\pm b_\phi$) be Δ if the antecedent of σ does not contain ϕ, and Δ, ϕ (Δ, b_ϕ) otherwise.

Definition A sequent τ *is a translation of* σ iff:

σ is $\Delta \pm \phi < \phi$	and τ is $B_\Delta \pm \phi < \phi$;
σ is $\Delta \pm \phi \leq \phi$	and τ is $B_\Delta \pm b_\phi < \phi$; or
σ is $\phi \preceq \psi$ or $\phi \prec \psi$	and τ is $\phi \prec \psi$.

We can now specify a way of translating an arbitrary set of premises S. Since we are ignoring weak identities, the first thing to do is to purge them from S. Let S^* be the result of removing all weak identities from S. For the purposes of translation, we must ensure that the witnessing constants can play the "hole-plugging" role we intend them to play. In particular, since b_ψ is an all-purpose hole-plugger for weak grounds of ψ, we'll add that b_ψ itself is a full weak ground of ψ, *i.e.*, $b_\psi \leq \psi$. Further, we may assume that any full ground that does not include ψ is a full ground of b_ψ; the idea here is that, if $\Delta \leq \psi$ or $\Delta < \psi$ (where $\psi \notin \Delta$), then Δ is itself an all-purpose "hole-plugger". To illustrate, consider the following application of CUT in PLG:

$$\frac{\phi \leq \phi \quad \chi \leq \chi \quad \Delta \leq \psi \quad \phi, \chi, \psi \leq \psi}{\phi, \chi, \Delta \leq \psi}$$

Notice that, if $\phi \notin \Delta$, the validity of this application of CUT shows that Δ can "plug any holes" left by the removal of ψ from the left-hand side of the major premise. Because such Δ's may serve as a hole-pluggers, we will assume that each of them is either identical to or fully grounds b_ψ. Thus, for all sequents in S which have the form $\Delta \leq \psi$ or $\Delta < \psi$ we'll add a sequent of the form $\Delta \leq b_\psi$. A first step in giving an adequate interpretation of weak grounding claims, then, is importing the witnessing constants and adding the claims that ensure that those constants can play their intended "hole-plugging" role:

Definition The *b-expansion* of a set of sequents S is the set containing exactly the following:

1. every member of S;
2. $b_\phi \leq \phi$, for all sentences ϕ occurring in any sequent in S; and
3. $\Delta \leq b_\phi$, for all sequents $\Delta \leq \phi$ or $\Delta < \phi$ in S ($\phi \notin \Delta$).

Finally, we choose particular translations for sequents σ in a given set of premises S.

Definition The *translation function* $\mathbf{T}$ is the function from sequents in the language of PLG to sequents in the language of LSG such that:

$$
\begin{array}{lcll}
\mathbf{T}(\Delta < \phi) & = & \Delta < \phi; & \\
\mathbf{T}(\phi \prec \psi) & = & \phi \prec \psi; & \\
\mathbf{T}(\phi \preceq \psi) & = & \phi \prec \psi; & \\
\mathbf{T}(\Delta \leq \phi) & = & \Delta < \phi & (\phi \notin \Delta)\text{; and} \\
\mathbf{T}(\Delta, \phi \leq \phi) & = & \Delta, b_\phi < \phi & (\phi \notin \Delta)
\end{array}
$$

Putting all of this together, the translation of a set of premises S ($\mathbf{T}(S)$) will be the result of applications of $\mathbf{T}$ to the members of the b-expansion of S^*:

$$\mathbf{T}(S) = \{\mathbf{T}(\sigma) | \sigma \text{ is a member of the } b\text{-expansion of } S^*\}$$

7 Identification Schemes

This translation scheme won't quite allow us to characterize derivability in PLG in terms of derivability in LSG. There are sets of sequents such that the translation of those sets has consequences normally derivable in LSG that the set does not have in PLG. For instance, when $\phi \neq \psi$,

$$\mathbf{T}(\{\phi \leq \psi, \psi \leq \phi\}) \vdash^{s-} \phi \prec \phi$$

but

$$\{\phi \leq \psi, \psi \leq \phi\} \nvdash \phi \prec \phi.$$

There is, however, an obvious interpretation in accord with the spirit of the present proposal that secures the consistency of $\phi \leq \psi$ with $\psi \leq \phi$: both sequents will be true when ϕ and ψ express the same fact. The above translation scheme goes wrong, in effect, by assuming that distinct sentences express distinct facts. This problem is easily remedied.

Definition An *identification scheme* is a function f from the set of sentences into the set of sentences such that $f(f(\phi)) = f(\phi)$.[21]

Clearly, if f is an identification scheme, the relation ϕ and ψ stand in just in case $f(\phi) = f(\psi)$ is an equivalence relation. Thus, an identification scheme partitions the set of sentences and chooses a representative for each partition. If Δ is a set of sentences, then I will often write '$f(\Delta)$' to indicate the image of Δ under f, and similar remarks apply to sequents and sets of sequents.

An identification scheme represents an hypothesis about which sentences state the same facts. How we should translate a weak sequent depends on the identities of the underlying facts. For instance, the proper translation of, *e.g.*, $\phi \leq \psi$ depends on whether ϕ and ψ report the same fact: if they do, then it is a weak identity and we ignore it; if they don't, then we translate it as a strict grounding claim. Call a translation of the result of the application of an identification scheme f to a sequent or a set of sequents *a translation under* f of the sequent or set. We know how to translate each sequent under each identification scheme. But we've got no information about which identification schemes get the identities of the facts right. How are we supposed to know how to translate the sequent?

We don't have to know how. The logic of ground is blind to which identification scheme is correct. Given an identification scheme, our translation together with LSG tells us what follows from $\phi \leq \psi$. So, we may take what follows from that weak sequent *independently of any identification scheme* to be what follows from some translation of that sequent *under every identification scheme.* Compare this to the case of disjunction. Suppose we're given as a premise the disjunction "it is either chilly or windy". The truth of the premise gives us no information about which disjunct is true. But this is no bar to determining what follows from the disjunction: the logical consequences of the disjunction are those claims that are logical consequences of *both* disjuncts. Similarly, the logical consequences of $\phi \leq \psi$ are those claims that follow from it no matter which facts are which. Since this sequent contains only two sentences, there are only two hypotheses about which facts are which that are relevant to its interpretation: either ϕ and ψ express the same fact, or they express distinct facts. In the former case, we interpret $\phi \leq \psi$ as the claim that ϕ is identical to ϕ (*i.e.* ψ). In the latter case, we interpret $\phi \leq \psi$ as the claim that ϕ strictly grounds

[21] Thanks to an anonymous referee for suggesting a significant refinement here.

ψ. Thus, our proposal in effect interprets $\phi \leq \psi$ as a disjunction: "either $\phi = \psi$ or ϕ strictly grounds ψ." The logical consequences of this disjunction, as with every disjunction, are the logical consequences of *both disjuncts*. Since sequents may contain infinitely many sentences, we can't use (finitary) disjunctions to translate all of the sequents in the language of PLG. But we will show, roughly, that what follows from a set of sequents S in PLG is given by what follows in LSG from a translation of S *under every identification scheme*.

8 Reducibility of $\vdash$ to $\vdash^s$

We started with two main desiderata on an interpretation of weak ground: (*i*) it should verify the principles of PLG; and (*ii*) it should allow us to reduce derivability in PLG to derivability in LSG. Our three main formal results show that these desiderata are satisfied. The first result shows that the interpretation verifies the inference rules of PLG:

Theorem F.1 *If $S \vdash \sigma$, then for all identification schemes f, $f(\sigma)$ either is a weak identity or has a translation τ such that $\mathbf{T}(f(S)) \vdash^s \tau$.*

In short, derivability in PLG is preserved under our translation scheme.

Two further formal results show that we can reduce derivability in PLG to derivability in LSG. The reduction proceeds in two steps. First, we show that an arbitrary sequent σ is derivable in PLG from a set of sequents S just in case either S is inconsistent (in PLG) or a translation of σ is normally derivable in LSG from a translation of S.

Theorem F.8 *$S \vdash \sigma$ iff either S is inconsistent in PLG, or, for all identification schemes f, if $f(\sigma)$ is not a weak identity, then it has a translation τ such that $\mathbf{T}(f(S)) \vdash^{s-} \tau$.*

This shows that derivability in PLG is reducible to derivability in LSG *and consistency in PLG*. We complete the reduction by showing that the consistency of S in PLG is reducible to derivability in LSG:

Theorem F.9 *S is inconsistent in PLG iff there is a sequent $\phi \prec \phi$ such that for all identification schemes f, $\mathbf{T}(f(S)) \vdash^{s-} f(\phi) \prec f(\phi)$.*

Thus, the pure logic of ground is reducible via our translation scheme to the logic of strict ground.

9 The Limits of the Reduction

This last theorem does not quite guarantee that when S is consistent there is an identification scheme on which the translation of S is also consistent. That would require there to be a *single* identification scheme f such that the translation of S under f is consistent. However, it is easy to give a simple general recipe for constructing such an identification scheme given a consistent set of sequents (see theorem F.10 below). Thus, if S is consistent (in PLG), then it has an interpretation in LSG that is consistent. In fact, S is consistent if and only if it has a consistent interpretation on some identification scheme.

In proving the reducibility of derivability in PLG to derivability in LSG, we show a subsidiary result concerning the logic of weak grounding claims:

Theorem F.3 ***($\leq$-Sufficiency)*** *Suppose σ is not a weak identity, but has the form $\Xi \leq \phi$, and let τ be a translation of σ. If $\mathbf{T}(S) \vdash^{s-} \tau$, then $S \vdash^{-} \sigma$.*

This result in effect shows that there is a particular identification scheme f – *viz.*, the trivial scheme on which $f(\phi) = \phi$ for all sentences ϕ – such that normal derivability of a translation of any full weak grounding claim from the translation of $f(S)$ guarantees normal derivability of that claim from S.[22] It would be nice if we could show something similar for strict sequents. Unfortunately, we can't. There is not generally any such identification scheme. Consider, for instance, the set of sequents

$$\{\phi_0 \leq \psi, \qquad \phi_1 \leq \psi, \qquad \phi_0 < \chi\}$$

Every identification scheme yields a translation of this set which implies in LSG some strict sequent not implied in PLG by the set itself. If we are to avoid implying $\phi_0 < \psi$, then we must identify ϕ_0 and ψ. If we are to avoid implying $\phi_1 < \psi$, then we must identify ϕ_1 and ψ. Finally, if we are to avoid implying $\phi_1 < \chi$, then we must avoid identifying ϕ_0 and ϕ_1. There is no identification scheme which can meet these constraints. This is exactly what we should expect on the interpretive proposal I have been exploring. According to the intuitive idea guiding our interpretation of weak grounding claims, $\phi_0 \leq \psi$ expresses in effect the disjunction, "either ϕ_0 is ψ or ϕ_0 (strictly) grounds ψ;" similarly, $\phi_1 \leq \psi$ expresses the disjunction, "either ϕ_1 is ψ or ϕ_1 (strictly) grounds ψ."

[22] Theorem F.4 shows a similar result for partial weak grounding claims.

The conjunction of these two claims with the claim that ϕ_0 grounds χ requires that at least one of

$$\phi_0 < \psi \qquad \phi_1 < \psi \qquad \phi_1 < \chi$$

be true, but does not require that any particular one of them be true.

In fact, there are consistent sets of sequents in PLG whose consistent interpretations in LSG have consequences that the original sets do not have. Consider, for instance, the set of sequents

$$\{\phi < \psi, \qquad \psi \preceq \chi, \qquad \chi \preceq \psi\}$$

We get a consistent translation only if our identification scheme f identifies ψ and χ. Otherwise, the last two sequents would be translated $f(\psi) \prec f(\chi)$ and $f(\chi) \prec f(\psi)$, respectively, and so the translation would be inconsistent. The translation under f of $\phi{<}\psi$ would therefore be identical to the translation under f of $\phi{<}\chi$. Thus, given the hypothesis about which sentences state the same facts required to make our interpretation of this set consistent, our interpretation of the set has a logical consequence that the set itself does not have. Supposing that that hypothesis fails, our interpretation of the set is inconsistent and so still has this consequence (via NON-CIRCULARITY). Thus, our interpretation of the set in question has a consequence that the set itself does not have.[23]

In this sense, our interpretation of sequents in PLG is not conservative. That is, the biconditional

$$S \vdash \sigma \text{ iff } (\forall f)\mathbf{T}(f(S)) \vdash^s \text{some translation of } f(\sigma)$$

fails in the right-to-left direction for some sequents σ expressing strict grounding claims. Our interpretation does, however, enjoy a qualified form of conservativity for consistent sets of sequents in PLG:[24]

Theorem F.7 *if S is consistent in PLG, then $S \vdash \sigma$ iff for all identification schemes f, $f(\sigma)$ either is a weak identity or has a translation τ such that $\mathbf{T}(f(S)) \vdash^{s-} \tau$.*

[23] Fine reports (personal correspondence) that in his view this inference is not valid. Presumably, then, he has in mind a notion of weak ground for which the interpretation I have offered is wrong. For the reasons stated in §2, it is not clear to me what that notion is.

[24] Conservativity holds for inconsistent sets. By theorem F.9, a set that is inconsistent in PLG gets an inconsistent translation under every identification scheme. It is a trivial consequence of this that the biconditional

$$S \vdash \sigma \text{ iff } (\forall f)\mathbf{T}(f(S)) \vdash^s \text{some translation of } f(\sigma)$$

holds if S is inconsistent.

Is the failure of the interpretation I have proposed to be conservative a reason to reject it? Certainly, there are limits on how far violations of conservativity may go. I argued in section 6, for instance, that an interpretation of $\phi, \psi, \chi \leq \chi$ on which it expresses the same grounding claim as $\phi, \psi \leq \chi$ goes too far because that interpretation identifies the two claims even though PLG does not sanction the derivability of the second from the first. This interpretation leaves us with an intolerable mystery: why does PLG not sanction the interderivability of these two claims, if they are synonymous, as the interpretation requires? This failure of conservativity goes too far.

Still, unqualified conservativity is not required in order for us to show that our interpretation of weak ground is adequate. After all, our interpretation uses logical resources, embedded in the apparatus of identification schemes, which are not independently representable in the language of PLG. PLG does not allow us to directly represent the fact that ϕ and ψ are distinct, for instance, without also representing a grounding relation between them. We should expect a richer representation of the relations among facts to yield consequences beyond those sanctioned by PLG. By way of analogy, recall that when we add identity to predicate logic, our ability to represent the facts is enriched. Thus, our logical representations of "Hesperus is shining" and "Hesperus = Phosphorus" get richer, in such a way that the representations of this pair of sentences in predicate logic with identity have a logical consequence in the original language – "Phosphorus is shining" – that their representations in predicate logic without identity do not have. Thus, we have an answer to the question of why PLG does not sanction the derivability of $\phi < \chi$ from

$$\{\phi < \psi, \qquad \psi \preceq \chi, \qquad \chi \preceq \psi\}.$$

PLG does not sanction this derivability because it does not have the logical resources to draw from the conjunction of $\psi \preceq \chi$ and $\chi \preceq \psi$ the consequence that $\psi = \chi$, and it does not have the logical resources to use that identity to substitute an occurrence of χ in for the occurrence of ψ in $\phi < \psi$. Our interpretation, on the other hand, uses a resource of this sort: the notion of an identification scheme.

There is a related further limitation of our proposed reduction of PLG to LSG. The application of our translation scheme to a weak sequent in PLG does not yield a definition in the language of LSG of that sequent. As I have emphasized, the interpretation of weak grounding claims I have offered is disjunctive: $\phi \leq \psi$ is, in effect, interpreted as the disjunctive claim that ϕ either strictly grounds ψ or is identical to it. Also, the interpretation of full weak grounding claims involves existential quantification: a sequent of the form $\Delta, \phi \leq \phi$ is in effect interpeted (*modulo* an identification scheme) as the claim that there are some facts Γ such that Δ, Γ strictly grounds ϕ. Neither disjunction nor existential quantification of the relevant sort are expressible in the language of LSG. Thus, the results of applying our translation scheme to weak sequents are strict sequents that can *represent* the weak grounding claims without actually *defining* them, in a manner analogous to the way in which Skolem normal forms can *represent* existentially quantified claims without actually defining them. As a result, there are things we can say in the language of PLG that we can't, strictly speaking, say in the language of LSG.[25]

How should this last limitation affect our assessment of the significance of the proposed interpretation? I have aimed to interpret weak grounding claims in terms of strict grounding claims while avoiding the problems encountered by the interpretations discussed in §2. Recall that Fine proposed to define weak ground in terms of strict ground. According to the definition, some facts weakly ground ϕ if and only if they strictly ground (perhaps in concert with some further facts Γ) all of the facts strictly grounded (perhaps in concert with Γ) by ϕ. Let's take Fine's proposal as a model. My proposal shares three features with Fine's. First, like Fine's proposal it offers an interpretation of weak sequents in terms of strict ground. So, given an antecedent grasp of the notion of strict ground, the interpretation helpfully specifies the meaning of a given weak sequent. Second, like Fine's proposal, the interpretation appeals to expressive resources – including quantification and truth-functions of grounding claims – that go beyond the language of LSG. Thus, my proposed translation scheme, like Fine's, does not map any weak sequents to any particular strict sequent in the language of LSG. Instead, my proposal maps a given weak sequent to a *family* of strict sequents, and the logical features of the weak sequent are analyzed by appeal to the logical features of that family. Third, like Fine's proposal, my proposal is clear enough that we can investigate its consequences for the logic of weak grounding claims by formal methods. An investigation

[25] Thanks to Kit Fine for bringing this limitation to my attention, and suggesting the comparison to Skolem normal forms.

of this sort has revealed that my interpretation validates all of the the axioms governing weak grounding claims, and, more generally, that derivability of a weak sequent in PLG is reducible to the derivability of its translations in LSG. So, the interpretation offered takes a step forward in understanding a notion of weak ground and explaining why that notion satisfies the axioms of PLG.

Still, it would be nice to have a one-for-one translation of weak sequents in a formally specified extension of the language of LSG. Obviously, the expressive resources missing from LSG are exactly those aspects of the interpretation that are implemented using witnessing constants and (quantification over) identification schemes: (infinitary) disjunction, fact identity, and quantification over facts. The specification of a system with those expressive resources and the demonstration of the adequacy of the proposed translation of weak sequents into that system must await separate treatment.

10 Weak Ground in the Semantics for LSG

There is one last respect in which the present attempt to expunge any notion of weak ground from the foundations of the pure logic of ground falls short. As Fine notes, his semantics give pride of place to the interpretation of weak full grounding claims, using the truth conditions for weak full grounding that we discussed in §2 above. The semantics says, in effect, that a full weak grounding claim of the form $\phi \leq \psi$ is true in a model iff every truthmaker for ϕ (according to the model) is also a truthmaker for ψ. The truth conditions for full strict ground are then defined on this basis. A full strict grounding claim of the form $\phi < \psi$ is true in a model iff ϕ (according to the model) is a full weak ground for ψ, and there are no facts that, together with ψ, compose a full weak ground for ϕ. The second clause in this definition is meant to exclude the degenerate case in which ψ itself is a full weak ground of ϕ: if in a model every truthmaker for ψ is also a truthmaker for ϕ, then $\phi < \psi$ is not true in the model.

This dependence on weak ground offers a reason for thinking that the semantics here proposed for LSG is inadequate to the notion of (strict) ground whose logical analysis is at issue. Suppose one last time that it is chilly but not sunny. According to Fine's semantics, its being chilly is a strict ground for its being either chilly or sunny only if there is a truthmaker for 'it's either chilly or sunny' that is not also a truthmaker for 'it's chilly.' Suppose again that truthmakers are sparse, so that the only parts of the actual world which are truthmakers for a true disjunction with a false disjunct are the truthmakers for its true disjunct. Our semantics will then imply the implausible claim that its being chilly is not a strict ground for its being either chilly or sunny. What's more, the truth condition thereby specified by the semantics need not even be construed as an *analysis* or *reduction* of the notion of strict ground. All that's needed to pose this problem is that we have specified a necessary and sufficient condition for the truth of strict grounding claims. If our model theoretic treatment of strict ground gets the truth conditions for strict grounding claims wrong, then that's reason for thinking that the model theoretic treatment is wrong.

It would be nice to have a more adequate semantics for strict ground. That project must be pursued on another occasion.[26] Still, we shouldn't lose sight of what has been shown. I have offered an interpretation of weak grounding claims that verifies the principles of PLG. We can use the interpretation to reduce derivability in PLG to derivability of the translation in LSG. Thus, we have a helpful way of understanding Fine's formalization of the logic of ground, including the logic of weak ground, that does not rely on a primitive notion of weak ground.[27]

Appendices

A Preliminaries

For convenience, I will sometimes call the rightmost premise of an instance of CUT its *major premise*, and the other premises *minor premises*. The *depth* of a derivation (in LSG or PLG) is defined recursively. Any derivation consisting of a single node (*i.e.* of a premise or single instance of identity) has a depth of 1. Otherwise, the depth of a derivation is the least ordinal greater than the depth of every immediate subderivation. Notice that infinite depths are allowed, since, *e.g.*, an instance of CUT in PLG may have infinitely many premises, and there

[26] I outline and defend a simple proposal in [deRosset, manuscript].

[27] The idea for this paper arose in correspondence with Jon Erling Litland. Thanks to Jon for his patience and insight. I also owe great thanks to Mark Moyer for extensive comments on a very rough early draft. Thanks also to Kit Fine for comments and discussion of a draft and some of the underlying ideas.

may be no finite upper bound on the depth of the premises. For instance, any derivation witnessing

$$\{\phi_0 \leq \phi_1, \qquad \phi_1 \leq \phi_2, \qquad ..., \qquad \phi_1, \phi_2, ... \leq \psi\} \quad \vdash \quad \phi_0 \leq \psi$$

will have infinite depth. Many of the proofs that follow deploy an induction on the depth of derivations.

B Chains

A useful tool for studying the logic of partial grounding claims is the notion of a *chain*.

Definition Let S be a set of sequents (either in PLG or in LSG). The notion of an *S-chain* is defined by recursion:

1. for each sequent $\phi, \dots \ll \psi$ in S, $\langle \phi, \psi \rangle$ is an S-chain;
2. if $\langle \phi, \phi_0, \phi_1, \dots, \chi \rangle$ is an S-chain and $\langle \chi, \chi_0, \chi_1, \dots, \psi \rangle$ is an S-chain, then $\langle \phi, \phi_0, \phi_1, ..., \chi, \chi_0, \chi_1, ..., \psi \rangle$ is an S-chain; and
3. nothing is an S-chain unless it is required to be so by clauses (1) and (2).

Intuitively, an S-chain is a sequence of sentences, where each adjacent pair in the sequence is such that the first element of the pair is at least a partial weak ground of the second element according to some claim in S.[28]

Definition Let S be a set of sequents in PLG. The notion of a *strict S-chain* is defined by recursion:

1. if $\langle \phi_0, \phi_1, ... \rangle$ is an S-chain and $\phi_0 \prec \phi_1$ or $\phi_0, \Delta < \phi_1$ are in S, then $\langle \phi_0, \phi_1, ... \rangle$ is a strict S-chain;
2. if $\langle \phi_1, \phi_2, ... \rangle$ is a strict S-chain and $\langle \phi_0, \psi_0, \psi_1, ..., \psi_n, \phi_1 \rangle$ is an S-chain, then $\langle \phi_0, \psi_0, \psi_1, ..., \psi_n, \phi_1, \phi_2, ... \rangle$ is a strict S-chain;
3. if $\langle \phi_0, \psi_0, \psi_1, ..., \phi_1 \rangle$ is a strict S-chain and $\langle \phi_1, \phi_2, ... \rangle$ is an S-chain, then $\langle \phi_0, \psi_0, ..., \phi_1, ... \rangle$ is a strict S-chain; and

[28]Fine introduces the notion of an S-chain at [Fine, 2012a, p. 12]. His definition is slightly different from the one offered here.

4. nothing is a strict S-chain unless it is required to be so by clauses (1)-(3).[29]

The burden of the next three results is to show that, intuitively, we may think of a strict S-chain as a sequence of sentences such that S implies that the first member of the sequence is a partial strict ground of every subsequent member. In what follows, when discussing S-chains I will omit the angle brackets '$\langle$' and '$\rangle$'.

Theorem B.1 *If $S \vdash^- \phi, ... \ll \psi$ and $\phi \neq \psi$ then there is an S-chain of the form $\phi, ..., \psi$. [Fine, 2012a, p. 12, Lemma 4.4]*

Proof The proof is by induction on the depth of the derivation witnessing $S \vdash^- \phi, ... \ll \psi$. The basis case and most of the induction steps are trivial. I'll just do the case of CUT for illustration. Suppose that a derivation witnessing the derivability of a sequent of the form $\phi, ... \ll \psi$ from S ends with a CUT of the form:

$$\frac{\Delta_0 \leq \phi_0 \quad \Delta_1 \leq \phi_1 \quad ... \quad \phi_0, \phi_1, ... \leq \psi}{\Delta_0, \Delta_1, ... \leq \psi} \quad ,$$

where $\phi \in \Delta_i$ for some minor premise $\Delta_i \leq \phi_i$.[30] The inductive hypothesis (henceforth "IH") gives us an S-chain $\phi, \chi_0, \chi_1, ..., \phi_i$ corresponding to this minor premise, and an S-chain $\phi_i, \xi_0, \xi_1, ..., \psi$ corresponding to the major premise. By the definition of *S-chain*, $\phi, \chi_0, \chi_1, ..., \phi_i, \xi_0, \xi_1, ..., \psi$ is an S-chain.
□

We can prove a similar result by similar means for LSG:

Theorem B.2 *If $S \vdash^{s-} \phi, ... < \psi$ or $S \vdash^{s-} \phi \prec \psi$ then there is an S-chain of the form $\phi, ..., \psi$.*

Theorem B.3 *$S \vdash^- \phi \prec \psi$ iff there is a strict S-chain of the form $\phi, ..., \psi$. [Fine, 2012a, p. 12, Lemmas 4.3, 4.4]*

[29] This definition of a strict S-chain differs from Fine's, which he offers at [Fine, 2012a, p. 12].

[30] From now on, I will say that a sequent σ "comes by" a rule or (an inference of a given form) to mean that there is a derivation witnessing the derivability of σ which ends with an application of that rule (or an inference of that form). For instance, the supposition I've just made could be expressed instead by, "Suppose $\phi, ... \ll \psi$ comes by CUT."

Proof $\Rightarrow$: We'll show the stronger claim that if $S \vdash^- \phi \prec \psi$ or $S \vdash^- \phi, ... < \psi$ then there is a strict S-chain of the form $\phi, ..., \psi$. The proof is by induction on the depth of derivations witnessing $S \vdash^- \phi \prec \psi$ or $S \vdash^- \phi, ... < \psi$. The basis case is trivial, and the induction steps are for the most part straightforward. I'll do the case of TRANSITIVITY($\preceq/\prec$) for illustration. Suppose $\phi \prec \psi$ comes by an application of TRANSITIVITY($\preceq/\prec$):

$$\frac{\phi \preceq \chi \quad \chi \prec \psi}{\phi \prec \psi}$$

IH gives us a strict S-chain $\chi, \theta, ..., \psi$. If $\phi = \chi$, then that strict S-chain suffices. Otherwise, theorem B.1 gives us an S-chain $\phi, ..., \chi$. Clause 2 of the definition of a *strict S-chain* then gives us the strict S-chain $\phi, \theta \, ..., \psi$.

$\Leftarrow$: Suppose that $\phi, ..., \psi$ is a strict S-chain. We'll show $S \vdash^- \phi \prec \psi$ by induction on the recursive definition of *strict S-chain*.

Basis case: Suppose that our strict S-chain has the form $\phi, \psi_0, \psi_1, ..., \psi_n, \psi$, and either $\phi \prec \psi_0$ or $\phi, ... < \psi_0$ is a member of S. By the definition of S-chain, there are sequents $\psi_i, ... \ll \psi_{i+1}$ and $\psi_n, ... \ll \psi$ in S. The application of SUBSUMPTION rules to each of these (other than $\phi \preceq \psi$) yields: $S \vdash \psi_i \preceq \psi_{i+1}$ (for each i) and $S \vdash \psi_n \preceq \psi$. Thus, repeated application of TRANSITIVITY($\prec/\preceq$) will suffice.

Induction step: Suppose that our strict S-chain has the form $\phi, \chi_0, \chi_1, ...\chi_n, \psi_0, ..., \psi$, where $\psi_0, ..., \psi$ is a strict S-chain and $\phi, \chi_0, \chi_1, ...\chi_n, \psi_0$ is an S-chain. As above, applications of SUBSUMPTION yield $S \vdash^- \phi \preceq \chi_0$, $S \vdash^- \chi_i \preceq \chi_{i+1}$ (for each i), and $S \vdash^- \chi_n \preceq \psi_0$. Thus, repeated applications of TRANSITIVITY($\preceq/\preceq$) give us $S \vdash^- \phi \preceq \psi_0$. IH gives us $S \vdash^- \psi_0 \prec \psi$, so an application of TRANSITIVITY($\preceq/\prec$) yields $S \vdash \phi \prec \psi$.

Suppose now that our strict S-chain has the form $\phi, ..., \psi_0, ..., \psi$, where $\phi, ..., \psi_0$ is a strict S-chain and $\psi_0, ..., \psi$ is a chain. IH gives us $S \vdash^- \phi \prec \psi_0$, and use of SUBSUMPTION and TRANSITIVITY($\preceq/\preceq$) in the same manner as above give us $S \vdash^- \psi_0 \preceq \psi$. Thus, an application of TRANSITIVITY($\prec/\preceq$) yields $S \vdash^- \phi \prec \psi$.

□

Theorem B.4 *Let S be a set of sequents of LSG. If there is a S-chain of the form $\phi, ..., \psi$, then $S \vdash^{s-} \phi \prec \psi$.*

Proof Suppose we're given a S-chain of the form $\phi, ..., \psi$. We'll prove the result by induction on the length of that chain.

Basis case: Suppose $\phi, ..., \psi$ has length 2. Then there is a sequent $\phi, ... < \psi$ or $\phi \prec \psi$ in S.

Induction step: Suppose that every chain of length less than n (where $n > 2$) is such that the result holds. Let $\phi, ..., \psi$ be a S-chain of length n. Then this chain has the form $\phi, ..., \chi, \psi$, where $\phi, ..., \chi$ is a S-chain of length $< n$. It's also easy to show that any final subsequence of a $T(S)$-chain is also a S-chain, so χ, ψ is a S-chain of length 2. IH yields $S \vdash^{s-} \phi \prec \chi$ and $S \vdash^{s-} \chi \prec \psi$. A single application of TRANSITIVITY yields the result.
□

Definition A *(strict) S-loop* is a (strict) S-chain of the form $\phi, ..., \phi$

An immediate corollary of theorem B.3 is that S is consistent in PLG just in case there are no strict S-loops.

Definition A *purely weak S-chain* is an S-chain such that, for every adjacent pair of sentences ϕ, ψ, S contains neither a sequent of the form $\phi, ... < \psi$, nor a sequent of the form $\phi \prec \psi$.

Another immediate corollary of theorem B.3 is that S is consistent in PLG just in case every S-loop is a purely weak S-loop.

C Model Theory, Soundness, and Completeness for LSG

Fine defines the notions of a *fact model* of PLG (and of a *generalized fact model*), and defines the notion of truth in a (generalized) fact model for sequents of PLG [Fine, 2012a, p. 8]. I will speak merely of "models" when discussing (generalized) fact models. Let $\mathcal{M}$ be a model of PLG; following Fine, I will write '$\mathcal{M} \vDash_{PLG} \sigma$' to mean that σ is a sequent true in $\mathcal{M}$. Fine also offers the natural definitions of the truth of a set of sequents S in a model, and of a sequent σ's being a consequence in PLG of S:

- a set S of sequents is *true in* a model $\mathcal{M}$ ($\mathcal{M} \vDash_{PLG} S$) iff $(\forall \sigma \in S)\mathcal{M} \vDash_{PLG} \sigma$; and

- a sequent σ is a *consequence of* a set of sequents S ($S \vDash_{PLG} \sigma$) iff for all models $\mathcal{M}$, if $\mathcal{M} \vDash_{PLG} S$ then $\mathcal{M} \vDash_{PLG} \sigma$.

We will use Fine's semantics to give a semantics for LSG of the obvious sort:

- $\mathcal{M}$ is a *model* of LSG iff $\mathcal{M}$ is a model of PLG.
- A sequent σ of LSG is *true in* a model $\mathcal{M}$ ($\mathcal{M} \vDash^s \sigma$) iff $\mathcal{M} \vDash_{PLG} \sigma$;
- a set S of sequents is *true in* a model $\mathcal{M}$ ($\mathcal{M} \vDash^s S$) iff $(\forall \sigma \in S)\mathcal{M} \vDash^s \sigma$; and
- a sequent σ is a *consequence of* a set of sequents S ($S \vDash^s \sigma$) iff for all models $\mathcal{M}$, if $\mathcal{M} \vDash^s S$ then $\mathcal{M} \vDash^s \sigma$.

We now show that LSG is sound and complete for this semantics.

Theorem C.1 ***(Soundness)*** *If* $S \vdash^s \sigma$*, then* $S \vDash^s \sigma$*.*

Proof Every rule in LSG is derivable in PLG [Fine, 2012a, pp. 6, 7], so

$$(S \vdash^s \sigma \Rightarrow S \vdash \sigma).$$

Fine proves soundness for PLG [Fine, 2012a, p. 10, Theorem 3.2], so

$$(S \vdash \sigma \Rightarrow S \vDash_{PLG} \sigma).$$

Finally, it is an immediate consequence of the definition of '$S \vDash^s \sigma$' that

$$S \vDash_{PLG} \sigma \Rightarrow S \vDash^s \sigma.$$

□

The proof of completeness is a little more involved. Let S be a set of sequents of LSG. We will prove two lemmas concerning S.

Lemma C.2 *If* $S \vdash \phi \prec \psi$*, then* $S \vdash^s \phi \prec \psi$*.*

Proof The result is an immediate consequence of theorems B.3 and B.4.
□

To prove the second lemma, it is convenient to use a modified version of CUT in PLG. Every application of the original version of CUT in PLG has the following form:

$$\frac{\Delta_0 \leq \phi_0 \quad \Delta_1 \leq \phi_1 \quad ... \quad \phi_0, \phi_1, ... \leq \phi}{\Delta_0, \Delta_1, ... \leq \phi}$$

Call such an application of CUT *pure* iff either $\phi \neq \phi_i$ for all i, or $\phi = \phi_i$ for all i. Our modified version of CUT in PLG is such that (i) side premises are allowed; (ii) no minor premises are identities; and (iii) every instance is pure. Thus, instances of the modified version of cut will have one of two forms:

$$\frac{\Delta_0 \leq \phi_0 \quad \Delta_1 \leq \phi_1 \quad ... \quad \phi_0, \phi_1, ..., \Gamma \leq \phi}{\Delta_0, \Delta_1, ... \leq \phi}$$

or

$$\frac{\Delta_0 \leq \phi \quad \Delta_1 \leq \phi \quad ... \quad \phi, \Gamma \leq \phi}{\Delta_0, \Delta_1, ..., \Gamma \leq \phi}$$

where, for all i, $\phi \neq \phi_i$ and $\Delta_i \neq \{\phi_i\}$. It is easy to see that anything derivable using the original version of CUT is also derivable by this modified version and vice versa. Instances of the modified version can be transformed into instances of the original version by applying IDENTITY in PLG to yield minor premises of the form $\gamma \leq \gamma$ for all $\gamma \in \Gamma$. Instances of the original version can be transformed into instances of the modified version in two steps: first, eliminate all minor premises that are identities (unless all minor premises are identities, in which case the major premise is also the conclusion); and second, split the result of the first step into two pure parts if it is impure.

Lemma C.3 *If* $S \vdash \Delta < \phi$*, then* $S \vdash^s \Delta < \phi$

Proof For the purposes of induction, it is useful to prove something stronger. Suppose σ is not an identity sequent, but is of either the form $\Delta \pm \phi \leq \phi$ or the form $\Delta < \phi$. We will show that if $S \vdash \sigma$ then $S \vdash^s \Delta < \phi$ by induction on the depth of the derivation witnessing $S \vdash \sigma$. The basis case is trivial.
Induction step: The cases in which σ comes by SUBSUMPTION($</\leq$), and REVERSE SUBSUMPTION are trivial. The case in which σ comes by NON-CIRCULARITY is handled trivially by an application of Lemma C.2. So, as usual, the most difficult case is CUT. Recall that there our modified version of CUT may take only two forms, so there are two cases to cover.

1. σ comes by a CUT of the form

$$\frac{\Delta_0 \leq \phi_0 \quad \Delta_1 \leq \phi_1 \quad ... \quad \phi_0, \phi_1, ..., \Gamma \leq \phi}{\Delta_0, \Delta_1, ... \leq \phi}$$

where $\phi \neq \phi_i$ for any i and none of the minor premises are identities. IH applied to the major premise gives that $S \vdash^s \phi_0, \phi_1, ..., (\Gamma \setminus \phi) < \phi$. IH applied to the minor premises gives that $S \vdash^s (\Delta_i \setminus \phi_i) < \phi_i$ for each i. Let Γ^+ be the set of sentences ϕ_i such that there is a minor premise of the form $\Delta_i, \phi_i \leq \phi_i$. Then the sequent $\phi_0, \phi_1, ..., (\Gamma \setminus \phi) < \phi$ that the application of IH to the major premise told us was derivable from S in LSG *also* has the form $\phi_0, \phi_1, ..., (\Gamma \setminus \phi), \Gamma^+ < \phi$. So, the following is an instance of CUT in LSG that yields the result:

$$\frac{(\Delta_0 \setminus \phi_0) < \phi_0 \quad (\Delta_1 \setminus \phi_1) < \phi_1 \quad ... \quad \phi_0, \phi_1, ..., (\Gamma \setminus \phi), \Gamma^+ < \phi}{\Delta_0, \Delta_1, ..., \Gamma \setminus \phi < \phi}$$

2. σ comes by a CUT of the form:

$$\frac{\Delta_0 \leq \phi \quad \Delta_1 \leq \phi \quad ... \quad \phi, \Gamma \leq \phi}{\Delta_0, \Delta_1, ..., \Gamma \leq \phi}$$

where none of the minor premises are identities. IH yields:

$$S \vdash^s (\Delta_0 \setminus \phi) < \phi, \qquad S \vdash^s (\Delta_1 \setminus \phi) < \phi, \qquad ...,$$

If $\Gamma \neq \{\phi\}$ (so that the major premise is not an identity $\phi \leq \phi$) then IH also yields $S \vdash^s (\Gamma \setminus \phi) < \phi$. Thus, an application of AMALGAMATION yields the result

$$S \vdash^s \Delta_0, \Delta_1, ..., \Gamma \setminus \phi < \phi$$

□

Notice that an immediate consequence of lemmas C.2 and C.3 is that PLG is a conservative extension of LSG.[31] This makes the proof of completeness easy.

Theorem C.4 ***(Completeness)*** *If $S \vDash^s \sigma$, then $S \vdash^s \sigma$.*

Proof Suppose σ has either the form $\Delta < \phi$ or $\phi \prec \psi$. It follows trivially from the definition of '$S \vDash^s \sigma$' that if $S \vDash^s \sigma$ then $S \vDash_{PLG} \sigma$. Further, Fine proves completeness for PLG [Fine, 2012a, p. 18, Theorem 6.4], so if $S \vDash_{PLG} \sigma$, then $S \vdash \sigma$. It remains only to show that if $S \vdash \sigma$ then $S \vdash^s \sigma$. But this is an immediate consequence of lemmas C.2 and C.3.

□

[31]Thanks to an anonymous referee for emphasizing this point.

D Partial ground

One of the nice features of both LSG and PLG is that it possible to study the consequences of a set of sentences for partial grounding claims in isolation, by, in effect, using SUBSUMPTION to generate a set of exclusively partial grounding glaims.

Definition For any set of sequents S, *the partial image of* S ($S^{\prec}$) is the smallest set P such that:

1. if $\phi \prec \psi \in S$, then $\phi \prec \psi \in P$;
2. if $\phi \preceq \psi \in S$, then $\phi \preceq \psi \in P$;
3. if $\Delta < \psi \in S$, then $(\forall \phi \in \Delta)\phi \prec \psi \in P$; and
4. if $\Delta \leq \psi \in S$, then $(\forall \phi \in \Delta)\phi \preceq \psi \in P$.

We can now show that the results of translating S allow the derivation of exactly the same partial grounding sequents as the results of translating $S^{\prec}$.

Theorem D.1 *A sequence of sentences is a (strict) S-chain iff it is a (strict) $S^{\prec}$-chain.*

Proof The theorem is a straightforward consequence of two facts: (i) there is a sequent of the form $\phi, ... \ll \psi \in S$ iff there is a sequent of either the form $\phi \prec \psi$ or the form $\phi \preceq \psi$ in $S^{\prec}$; and (ii) there is a sequent of the form $\phi, ... < \psi$ or $\phi \prec \psi$ in S iff there is a sequent of the form $\phi \prec \psi \in S^{\prec}$.

An immediate corollary of theorems B.2 and B.4 is that $\mathbf{T}(S) \vdash^{s-} \phi \prec \psi$ iff there is a $\mathbf{T}(S)$-chain $\phi, ..., \psi$. Now we'll prove a similar result for our translation of the partial image of S.

Theorem D.2 *If $\mathbf{T}(S^{\prec}) \vdash^{s-} \phi \prec \psi$, then there is a $\mathbf{T}(S)$-chain of the form $\phi, ..., \psi$.*

Proof We'll use an induction on the depth of the derivation witnessing $\mathbf{T}(S^{\prec}) \vdash^{s-} \phi \prec \psi$.

Basis case: Suppose $\phi \prec \psi \in \mathbf{T}(S^{\prec})$. If $\phi = \psi$, then $\phi \prec \phi \in S^{\prec}$. So either $\phi \prec \phi \in S$ or $\phi ... < \phi \in S$. In either case, $\mathbf{T}(S)$ contains a sequent $\phi, ... \ll \psi$. If

$\phi \neq \psi$, then either $\phi \prec \psi \in S^{\prec}$ or $\phi \preceq \psi \in S^{\prec}$. So, some sequent of the form $\phi, ... \ll \psi \in S$, and thus some sequent of the form $\phi, ... \ll \psi \in \mathbf{T}(S)$.

Induction Step: Suppose that $\phi \prec \psi$ is derivable from $\mathbf{T}(S^{\prec})$ by TRANSITIVITY from $\phi \prec \chi$ and $\chi \prec \psi$. IH yields $\mathbf{T}(S)$-chains $\phi, ..., \chi$ and $\chi, ..., \psi$, so the result is immediate

□

The next theorem shows that we can squeeze witnessing constants out of the middle of $\mathbf{T}(S)$-chains.

Theorem D.3 *If there is a $\mathbf{T}(S)$-chain of the form $\phi, ..., \psi$, then there is a $\mathbf{T}(S)$-chain of the form $\phi, ..., \psi$ which contains no witnessing constants between ϕ and ψ.*

Proof Witnessing constants do not appear on both sides of any member of $\mathbf{T}(S)$. If there is a witnessing constant on the left of some sequent $\sigma \in \mathbf{T}(S)$ then σ has the form $\Delta, b_\chi < \chi$ (where Δ may be empty). If there is one on the right, then σ is $\Delta < b_\chi$. So, no $\mathbf{T}(S)$-chain contains two witnessing constants in a row. Let

$$\phi, ..., \chi_0, b_\chi, \chi_1, ..., \psi$$

be a $\mathbf{T}(S)$-chain. Then there is a sequent of the form $\Delta, \chi_0 < b_\chi \in \mathbf{T}(S)$ and a sequent of the form $\Gamma, b_\chi < \chi_1 \in \mathbf{T}(S)$. Because the latter is in $\mathbf{T}(S)$, $\chi = \chi_1$. Because the former is in $\mathbf{T}(S)$, there is also a sequent of the form $\Delta, \chi_0 < \chi \in \mathbf{T}(S)$. Thus, χ_0, χ_1 is a $\mathbf{T}(S)$-chain, which suffices to show that $\phi, ..., \chi_0, \chi_1, ..., \psi$ is a $\mathbf{T}(S)$-chain.

□

We can also almost always remove them from the ends of $\mathbf{T}(S)$-chains.

Theorem D.4 *(*i*) If there is a $\mathbf{T}(S)$-chain of the form $\phi, ..., b_\psi$, then there is a $\mathbf{T}(S)$-chain of the form $\phi, ..., \psi$; and (*ii*) If there is a $\mathbf{T}(S)$-chain of the form $b_\phi, ..., \psi$ and either $\phi \neq \psi$ or the length of the chain is greater than 2, then there is a $\mathbf{T}(S)$-chain of the form $\phi, ..., \psi$.*

Proof (*i*) follows straightforwardly from the fact that for every sentence $\psi \in S^*$, $b_\psi < \psi \in \mathbf{T}(S)$. (*ii*) follows from the fact that witnessing constants appear on the left of sequents in $\mathbf{T}(S)$ only if those sequents have the form $\Delta, b_\chi < \chi$ (where Δ may be empty).

Theorem D.5 *If there is a* $\mathbf{T}(S)$*-chain of the form* $\phi, ..., \psi$ *(where* ψ *is not a witnessing constant), then* $\mathbf{T}(S^{\prec}) \vdash^{s-} \phi \prec \psi$*.*

Proof Suppose we're given a $\mathbf{T}(S)$-chain of the form $\phi, ..., \psi$. By theorem D.3, we may assume that there are no witnessing constants between ϕ and ψ. We'll prove the result by induction on the length of that chain.

Basis case: Assume our $\mathbf{T}(S)$-chain $\phi, ..., \psi$ has length 2, *i.e.* the chain has the form ϕ, ψ. If ϕ is a witnessing constant, then $\phi = b_\psi$; and, since $b_\psi \leq \psi$ is in the b-expansion of S, $b_\psi < \psi \in \mathbf{T}(S)$. If $\phi = \psi$, then there is a sequent $\phi, ... < \psi$ or $\phi \prec \psi \in S$, and so that same sequent will be in $\mathbf{T}(S)$. If $\phi \neq \psi$, then there is a sequent $\phi, ... \ll \psi \in (S)$, and so either $\phi \preceq \psi$ or $\phi \prec \psi$ is in $S^{\prec}$.

Induction Step: Suppose that every chain of length less than n (where $n > 2$) is such that the result holds. Let $\phi, ..., \psi$ have length n. Then, for some χ, $\phi, ..., \psi$ has the form $\phi, ..., \chi, \psi$. IH gives us $\mathbf{T}(S^{\prec}) \vdash^{s-} \phi \prec \chi$. Since χ, ψ is also a $\mathbf{T}(S)$-chain, IH also gives us $\mathbf{T}(S^{\prec}) \vdash^{s-} \chi \prec \psi$. A single application of TRANSITIVITY yields the result.
□

Finally, we show that the result of applying an identification scheme to a partial image of a set S of sequents is the same as partial image of the result of applying the identification scheme to S.

Theorem D.6 *For all identification schemes* f*,* $f(S^{\prec}) = f(S)^{\prec}$*.*

Proof $\Rightarrow$: Suppose $\sigma \in f(S^{\prec})$. There are four cases:

(i) σ is $f(\phi) \prec f(\psi)$ for some ϕ, ψ such that $\phi \prec \psi \in S^{\prec}$, and either

- (a) $\phi, ... < \psi \in S$ or
- (b) $\phi \prec \psi \in S$.

(ii) σ is $f(\phi) \preceq f(\psi)$ for some ϕ, ψ such that $\phi \preceq \psi \in S^{\prec}$, and either

- (a) $\phi, ... \leq \psi \in S$ or
- (b) $\phi \preceq \psi \in S$.

In (ia): $f(\phi, ... < \psi) \in f(S)$. Thus, $f(\phi), ... < f(\psi) \in f(S)$, so $f(\phi) \prec f(\psi) \in f(S)^{\prec}$. In ($ib$): $f(\phi \prec \psi) \in f(S)$. Thus, $f(\phi) \prec f(\psi) \in f(S)$, so $f(\phi) \prec f(\psi) \in f(S)^{\prec}$.

Similar arguments handle cases (iia) and (iib).

$\Leftarrow$: Suppose $\sigma \in f(S)^{\prec}$. There are four cases:

(i) σ is $\phi \prec \psi$, and either

  (a) $\phi, \phi_0, \phi_1, ... < \psi \in f(S)$ or

  (b) $\phi \prec \psi \in f(S)$.

(ii) σ is $\phi \preceq \psi$, and either

  (a) $\phi, \phi_0, \phi_1, ... \leq \psi \in f(S)$ or

  (b) $\phi \preceq \psi \in f(S)$.

In (ia): There are sentences $\phi^*, \phi_0^*, \phi_1^*, ..., \psi^*$ such that

$$f(\phi^*) = \phi, \qquad f(\phi_0^*) = \phi_0, \qquad f(\phi_1^*) = \phi_1, \qquad ..., f(\psi^*) = \psi$$

and $\phi^*, \phi_0^*, \phi_1^*, ... < \psi^* \in S$. So, $\phi^* \prec \psi^* \in S^{\prec}$, and thus $f(\phi^*) \prec f(\psi^*)$ (*i.e.* $\phi \prec \psi) \in f(S^{\prec})$. In (ib): There are sentences ϕ^*, ψ^* such that

$$f(\phi^*) = \phi, \qquad f(\psi^*) = \psi$$

and $\phi^* \prec \psi^* \in S$. So, $f(\phi^*) \prec f(\psi^*) \in f(S^{\prec})$.

Similar arguments handle cases (iia) and (iib).

□

Theorem D.7 *For all identification schemes f, if ϕ and ψ are not witnessing constants, then*

$$\mathbf{T}(f(S)) \vdash^{s-} \phi \prec \psi \text{ iff } \mathbf{T}(f(S^{\prec})) \vdash^{s-} \phi \prec \psi.$$

This theorem is an immediate upshot of theorems B.2, B.4, D.2, D.5, and D.6.

E Witnessing constants in derivations

We can prove two results that are useful for dealing with witnessing constants in derivations in LSG. The first result is useful for eliminating witnessing constants from the left-hand side of a sequent.

Theorem E.1 ***(Factorization Lemma)*** *Suppose ϕ is not a witnessing constant. If $\mathbf{T}(S) \vdash^{s-} \Delta < \phi$ and $b_\phi \notin \Delta$, then there is a Δ^* such that $\Delta^* \subseteq \Delta$ and $\mathbf{T}(S) \vdash^{s-} \Delta^* < b_\phi$.*

Proof We'll use an induction on the depth of the derivation witnessing $\mathbf{T}(S) \vdash^{s-} \Delta < \psi$.

Basis case: $\Delta < \phi \in \mathbf{T}(S)$. Then $\Delta < b_\phi \in \mathbf{T}(S)$.

Induction step: Suppose $\Delta < \phi$ has the form $\Delta_0, \Delta_1, ... < \phi$ and comes by AMALGAMATION:

$$\frac{\Delta_0 < \phi \quad \Delta_1 < \phi \quad ...}{\Delta_0, \Delta_1, ..., < \phi}$$

IH gives us, for each i a nonempty Δ_i^* such that $\mathbf{T}(S) \vdash^{s-} \Delta_i^* < b_\phi$. Let $\Delta^* = \Delta_0^* \cup \Delta_1^*, ...$

Suppose $\Delta < \phi$ has the form $\Gamma, \Delta_0, \Delta_1, ... < \phi$ and comes by CUT:

$$\frac{\Delta_0 < \phi_0 \quad \Delta_1 < \phi_1 \quad ... \quad \Gamma, \phi_0, \phi_1, ... < \phi}{\Gamma, \Delta_0, \Delta_1, ... < b_\phi}$$

If $b_\phi = \phi_i$ for some i, then one of the minor premises has the form $\Delta_i < b_\phi$ and we're done. Otherwise, IH gives us some subset Γ^* of Γ and some subset $\{\phi_0^*, \phi_1^*, ...\}$ of $\{\phi_0, \phi_1, ...\}$ such that $\Gamma^* \cup \{\phi_0^*, \phi_1^*, ...\}$ is non-empty and $\mathbf{T}(S) \vdash^{s-} \Gamma^* \phi_0^*, \phi_1^*, ... < b_\phi$. Let

$$\Delta_0^* < \phi_0^*, \qquad \Delta_1^* < \phi_1^*, \qquad ...$$

be the minor premises $\Delta_i < \phi_i$ such that $\phi_i \in \{\phi_0^*, \phi_1^*, ...\}$. We have, then, the following CUT in LSG:

$$\frac{\Delta_0^* < \phi_0^* \quad \Delta_1^* < \phi_1^* \quad ... \quad \Gamma^*, \phi_0^*, \phi_1^*, ... < b_\phi}{\Gamma^*, \Delta_0^*, \Delta_1^*, ... < b_\phi}$$

It is easy to verify that $\Gamma^* \cup \Delta_0^* \cup \Delta_1^*, ...$ is a subset of $\Gamma \cup \Delta_0 \cup \Delta_1, ...$
□

Theorem E.1 says we may use CUT in LSG to eliminate a witnessing constant b_ϕ from the left-hand side of a sequent if we are given a premise $\Delta < \phi$ whose left-hand side does not contain b_ϕ. But the application of CUT leaves only a subset of Δ in place of b_ϕ. This next result shows in effect that we may replace b_ϕ with all of Δ, rather than just a subset.

Theorem E.2 *If* $\mathbf{T}(S) \vdash^{s-} \Gamma, b_{\psi_0}, b_{\psi_1}, ... < \psi$ *and*

$$\mathbf{T}(S) \vdash^{s-} \Xi_0 < \psi_0, \qquad \mathbf{T}(S) \vdash^{s-} \Xi_1 < \psi_1, \qquad , ...,$$

then $\mathbf{T}(S) \vdash^{s-} \Gamma, \Xi_0, b_{\psi_0}, \Xi_1, b_{\psi_1}, ... < \psi$.

Proof Suppose we're given a derivation d witnessing $\mathbf{T}(S) \vdash^{s-} \Gamma, b_{\psi_0}, b_{\psi_1}, ...<\psi$, and we're given $\mathbf{T}(S) \vdash^{s-} \Xi_i < \psi_i$ for each i. As a notational convention, let Ξ_χ designate Ξ_i if $\chi = \psi_i$. We'll show by induction that every subderivation of any derivation d witnessing $\mathbf{T}(S) \vdash^{s-} \Gamma, b_{\psi_0}, b_{\psi_1}, ...<\psi$ has the following property: if the conclusion of the subderivation is a sequent σ of the form $\Delta, b_{\psi'_0}, b_{\psi'_1}, ... < \phi$, where $b_{\psi'_0}, b_{\psi'_1}, ...$ are exactly the witnessing constants on the left-hand side of σ in $\{b_{\psi_0}, b_{\psi_1}, ...\}$, then $\mathbf{T}(S) \vdash^{s-} \Delta, \Xi_{\psi'_0}, b_{\psi'_0}, \Xi_{\psi'_1}, ... < \phi$. The strategy of the proof is pretty obvious: we find all of the places in the derivation where a relevant witnessing constant b_{ψ_i} first appears on the left. There we dump Ξ_i into the left-hand side by an application of AMALGAMATION. Then we make sure to schlep Ξ_i along though the rest of the derivation, reserving it as a set of side-premises in any applications of CUT.

Basis case: σ is a premise in d. Since σ contains a witnessing constant among $b_{\psi_0}, b_{\psi_1}, ...$, σ has the form $\Delta, b_{\psi_i} < \psi_i$. By hypothesis, $\mathbf{T}(S) \vdash^{s-} \Xi_i < \psi_i$. An application of AMALGAMATION gives us $\mathbf{T}(S) \vdash^{s-} \Delta, \Xi_i, b_{\psi_i} < \psi_i$.

Induction step: Suppose σ comes by AMALGAMATION in a subderivation of d:

$$\frac{\Delta_0, B_0 < \phi \quad \Delta_1, B_1 < \phi \quad ...}{\Delta_0, \Delta_1, ..., B_0, B_1, ... < \phi}$$

where, for each i, B_i is the set of exactly the witnessing constants in $\{b_{\psi_0}, b_{\psi_1}, ...\}$ in the antecedent of the relevant premise. IH gives us, for each premise $\Delta_j, B_j < \phi$, $\mathbf{T}(S) \vdash^{s-} \Delta_j, \Xi^*_j, B_j < \phi$, where $\Xi^*_j = \bigcup\{\Xi_i | b_{\psi_i} \in B_j\}$. Thus, an application of AMALGAMATION yields the result.

Suppose σ comes by CUT in a subderivation of d:

$$\frac{\Delta_0, B_0 < \phi_0 \quad \Delta_1, B_1 < \phi_1 \quad ... \quad \Gamma, \phi_0, \phi_1, ... < \phi}{\Gamma, \Delta_0, \Delta_1, ..., B_0, B_1, ... < \phi}$$

where, as above, for each i, B_i is the set of relevant witnessing constants. IH gives us for each minor premise $\Delta_j, B_j<\phi_j$, $\mathbf{T}(S) \vdash^{s-} \Delta_j, \Xi^*_j, B_j<\phi$, where $\Xi^*_j = \bigcup\{\Xi_i | b_{\psi_i} \in B_j\}$. Let B be exactly the witnessing constants in $\{b_{\psi_o}, b_{\psi_1}, ...\}$ in the antecedent of the major premise. IH gives us $\mathbf{T}(S) \vdash^{s-} \Gamma, \phi_o, \phi_1, ..., \Xi^* < \phi$, where $\Xi^* = \bigcup\{\Xi_i | b_{\psi_i} \in B\}$. Thus, an application of CUT gives us

$$\mathbf{T}(S) \vdash^{s-} \Gamma, \Xi^*, \Delta_0, \Xi_0^*, B_0, \Delta_1, \Xi_1^*, B_1, ... < \phi.$$

It is easy to verify that

$$\bigcup\{\Xi^*, \Xi_0^*, \Xi_1^*, ...\} = \bigcup\{\Xi_0, \Xi_1, ...\}.$$

□

F Correpondence of $\vdash$ and $\vdash^s$

Theorem F.1 *If $S \vdash \sigma$, then for all identification schemes f, $f(\sigma)$ either is a weak identity or has a translation τ such that $\mathbf{T}(f(S)) \vdash^s \tau$.*

Proof Fix f. We'll prove the result by induction on the depth of any derivation d witnessing $S \vdash \sigma$. The basis case is trivial. Suppose, then, that the result holds for all proper subderivations of d. We'll show it also holds for d. The case where σ comes by CUT is by far the hairiest. Most of the other cases are straightforward. I'll do SUBSUMPTION($\leq/\preceq$), and REVERSE SUBSUMPTION for illustration. Then I'll finish with CUT.

SUBSUMPTION($\leq/\preceq$): Suppose that σ is $\phi \preceq \psi$ and comes by an inference of the form:

$$\frac{\phi, \phi_0, \phi_1, ... \leq \psi}{\phi \preceq \psi}$$

If $f(\phi) = f(\psi)$, then $f(\sigma)$ is a weak identity, and we're done. Suppose, then, that $f(\phi) \neq f(\psi)$. IH gives us $\mathbf{T}(f(S)) \vdash^S \tau$ for τ some translation of $f(\phi, \phi_0, \phi_1, ... \leq \psi)$. Recall that $\mathbf{T}(f(S))$ applies the translation function $\mathbf{T}$ to the b-expansion of the result $f(S)^*$ of removing all weak identities from $f(S)$. Thus our translation τ has either the form $f(\phi), ... < f(\psi)$ or the form $b_{f(\phi)}, ... < f(\psi)$. So, theorem B.2 gives us either a $\mathbf{T}(f(S))$-chain $f(\phi), ..., f(\psi)$ or a $\mathbf{T}(f(S))$-chain $b_{f(\phi)}, ..., f(\psi)$. In the latter case, theorem D.4 gives us $\mathbf{T}(f(S))$-chain $f(\phi), ..., f(\psi)$. Thus, theorem B.4 implies $\mathbf{T}(f(S) \vdash f(\phi) \prec f(\psi)$.

REVERSE SUBSUMPTION: Suppose that σ is $\phi_0, \phi_1, ... < \phi$ and comes by an inference of the form:

$$\frac{\phi_0, \phi_1, ... \leq \phi \quad \phi_0 \prec \phi \quad \phi_1 \prec \phi \quad ...}{\phi_0, \phi_1, ... < \phi}$$

If $f(\phi) = f(\phi_i)$ for any i, then the translation under f of one of the premises $f(\phi_i \prec \phi)$ also has the form $f(\phi) \prec f(\phi)$, and IH and an application of NON-CIRCULARITY in LSG suffices.[32] Suppose, then, that $f(\phi) \neq f(\phi_i)$ for any i. IH gives us $\mathbf{T}(f(S)) \vdash^S \tau$ for τ some translation of $f(\phi_0, \phi_1, ... \leq \phi)$. So, τ has the form $B < \psi$, where B is a b-version of $\{f(\phi_0), f(\phi_1), ...\}$, and $\psi \notin \{f(\phi_0), f(\phi_1), ...\}$. Thus, τ is also a translation of $f(\phi_0, \phi_1, ... < \phi)$.

CUT: Suppose, finally, that σ is $\Delta_0, \Delta_1, ... \leq \phi$ and comes by an inference of the form:

$$\frac{\Delta_0 \leq \phi_0 \quad \Delta_1 \leq \phi_1 \quad ... \quad \phi_0, \phi_1, ... \leq \phi}{\Delta_0, \Delta_1, ... \leq \phi}$$

IH gives us: (i) for each minor premise $\Delta_i \leq \phi_i$ that is not a weak identity, a translation τ_i of $f(\Delta_i) \leq f(\phi_i)$ such that $\mathbf{T}(f(S)) \vdash^s \tau_i$, and (ii) a translation τ of $f(\phi_0, \phi_1, ... \leq \phi)$ such that $\mathbf{T}(f(S)) \vdash^s \tau$. (We're assuming that $f(\phi_0, \phi_1, ... \leq \phi)$ is not a weak identity, since otherwise we could apply AMALGAMATION to the translations τ_i of the minor premises to get a translation of $f(\sigma)$, and we would be done.) If ϕ is a sentence, let an occurrence of $\phi^{\pm b}$ in a specification of the form of a sequent or a set of sequents stand for either ϕ itself or its witnessing constant b_ϕ. Thus, τ has the form $f(\phi_0)^{\pm b}, f(\phi_0)^{\pm b}, ..., < f(\phi)$. Let

$$A = \{\tau_i | f(\Delta_i) \neq \{\phi_i\}\}.$$

Intuitively, A is the set of translations of non-identity premises (under f) in the cut: we may think of these as the "active" premises in the cut, since identity premises cannot be used in an application of CUT to add or remove sentences from the left-hand side of a sequent. (If A is empty, then τ is a translation of $f(\sigma)$ and we're done.) Let

$$\Gamma = \{f(\phi_i)^{\pm b} | f(\phi_i)^{\pm b} \text{ is in the left-hand side of } \tau \text{ and } \tau_i \notin A\}$$

Γ will be a set of side sentences for CUT in LSG. A has two (possibly overlapping) subsets W and M:

$$W = \{\tau_i \in A | b_{f(\phi_i)} \text{ is in the left-hand side of } \tau\}$$

[32]This is the only place in the proof of this theorem where a normal derivation in PLG of a sequent corresponds to a non-normal derivation in LSG of a translation of that sequent.

$$M = \{\tau_i \in A | f(\phi_i) \text{ is in the left-hand side of } \tau\}.$$

Intuitively, W is the set of translations of premises whose right-hand side $f(\phi_i)$ may fail to be matched by an occurrence of $f(\phi_i)$ in the left-hand side of τ: it's the set of premise translations suffering from a potential "witness mistmatch" problem. Finally, let $\Delta_0^W < \phi_0^W, \Delta_1^W < \phi_1^W, ...$ be exactly the members of W, and $\Delta_0^M < \phi_0^M, \Delta_1^M < \phi_1^M, ...$ be exactly the members of M. τ has the form:

$$\Gamma, \phi_0^M, \phi_1^M, ..., b_{\phi_0^W}, b_{\phi_1^W}, ... < f(\phi).$$

We'll do two applications of CUT in LSG, and show that the result is a translation of σ. First, let τ^* be the conclusion of the following application of CUT in LSG:

$$\frac{\Delta_0^M \leq \phi_0^M \quad \Delta_1^M \leq \phi_1^M \quad ... \quad \Gamma, \phi_0^M, \phi_1^M, ..., b_{\phi_0^W}, b_{\phi_1^W}, ... < f(\phi)}{\Gamma, \Delta_0^M, \Delta_1^M, ..., b_{\phi_0^W}, b_{\phi_1^W}, ... < f(\phi) (= \tau^*)}$$

Recall that IH gives us

$$\mathbf{T}(f(S) \vdash^s \Delta_0^w < \phi_0^W, \qquad \mathbf{T}(f(S) \vdash^s \Delta_1^w < \phi_1^W, \qquad ...$$

So, an application of theorem E.2 to τ^* gives us

$$\mathbf{T}(f(S) \vdash^s \Gamma, \Delta_0^M, \Delta_1^M, ..., \Delta_0^W, b_{\phi_0^W}, \Delta_1^W, b_{\phi_1^W}, ... < f(\phi).$$

It remains only to remove witnessing constants $b_{\phi_i^W}$ such that ϕ_i^W is not in the left-hand side of any translation of $f(\sigma)$. Let ϕ_i^W be a sentence such that $b_{\phi_i^W}$ is in the left-hand side of τ^*, but there is no ψ in the left-hand side of σ such that $f(\psi) = \phi_i^W$. Then Δ_i^W contains neither ϕ_i^W nor $b_{\phi_i^W}$, so theorem E.1 applies to give us a subset Δ_i^* of Δ_i^W such that $T(f(S)) \vdash^s \Delta_i^* < b_{\phi_i^W}$. Thus, the following application of CUT in LSG suffices:

$$\frac{\Delta_0^* < b_{\phi_0^W} \quad \Delta_1^* < b_{\phi_1^W} \quad ... \quad \Gamma, \Delta_0^M, \Delta_1^M, ..., \Delta_0^W, b_{\phi_0^W}, \Delta_1^W, b_{\phi_1^W}, ... < f(\phi)}{\Gamma, \Delta_0^M, \Delta_1^M, ..., \Delta_0^W, \Delta_0^*, \Delta_1^W, \Delta_1^*, ... < f(\phi)}$$

It is easy to verify that the left-hand side of the conclusion of this cut contains exactly the members of left-hand sides of the translations τ_i of $f(\Delta_i) \leq f(\phi_i)$. Thus, the conclusion of this last application of CUT in LSG is a translation of $f(\Delta_0, \Delta_1, ... \leq \phi)$.

□

The result cannot be improved to show that anything normally derivable from S in PLG is normally derivable from $T(S)$ in LSG. In footnote 32 I flagged the one place in the proof of this theorem where an appeal is made to NON-CIRCULARITY in LSG when there was no corresponding application of NON-CIRCULARITY in PLG. This use of a non-normal derivation in LSG is essential. For instance, there is a normal derivation in PLG of $\phi < \phi$ from $\phi \prec \phi$, using IDENTITY and REVERSE SUBSUMPTION. There is no corresponding normal derivation in LSG. It is obvious that any such case will be one in which S itself is inconsistent in PLG. So, the proof of F.1 also establishes:

Theorem F.2 *if S is consistent in PLG, then $S \vdash \sigma$ only if for all identification schemes f, $f(\sigma)$ is either a weak identity or has a translation τ such that $\mathbf{T}(f(S)) \vdash^{s-} \tau$.*

Theorem F.3 ***($\leq$-Sufficiency)*** *Suppose σ is not a weak identity, but has the form $\Xi \leq \phi$, and let τ be a translation of σ. If $\mathbf{T}(S) \vdash^{s-} \tau$, then $S \vdash^{-} \sigma$.*

Proof We'll assume that σ does not have the form $\phi \leq \phi$, since in that case the result is trivial. We'll prove the result by induction on the depth of any derivation d witnessing $T(S) \vdash^{s-} \tau$. For the purposes of induction, we'll show the stronger result that if $\mathbf{T}(S) \vdash^{s-} \tau$ and τ has the form $B_\Gamma < \phi^{\pm b}$ where B_Γ is a b-version of Γ, then $S \vdash^{-} \Gamma \leq \phi$.

Basis Case: Suppose $\tau \in \mathbf{T}(S)$ and has the form $B_\Gamma < \phi^{\pm b}$. Then τ is a translation of some sequent in the b-expansion of S^*, which results from removing all weak identities from S. If τ is a translation of a member of S^*, then we're done. If τ has the form $\Delta < b_\phi$, then either $\Delta < \phi$ or $\Delta \leq \phi$ is a member of S^* and we're done.

Induction step: Suppose τ comes by an application of AMALGAMATION of the following form:

$$\frac{\Xi_0 < \phi^{\pm b} \quad \Xi_1 < \phi^{\pm b} \quad \ldots}{\Xi < \phi^{\pm b} (= \tau)}$$

IH gives, for each of the premises a sequent $\Xi_i^* \leq \phi$ such that $S \vdash^{-} \Xi_i^* \leq \phi$ and Ξ_i is a b-version of Ξ_i^*. Fine shows that

$$\textbf{AMALGAMATION}(\leq)\text{:} \quad \frac{\Delta_0 \leq \phi \qquad \Delta_1 \leq \phi \qquad \ldots}{\Delta_0, \Delta_1, \ldots \leq \phi}$$

is valid in PLG. Thus, the following application of AMALGAMATION($\leq$) in PLG suffices:

$$\frac{\Xi_0^* \leq \phi \quad \Xi_1^* \leq \phi \quad ...}{\Xi^* \leq \phi}$$

Suppose τ comes by an application of CUT of the following form:

$$\frac{\Xi_0 < \phi_0^{\pm b} \quad \Xi_1 < \phi_1^{\pm b} \quad ... \quad \Gamma, \phi_0^{\pm b}, \phi_1^{\pm b}, ... < \phi^{\pm b}}{\Gamma, \Xi_0, \Xi_1, ... < \phi^{\pm b}}$$

By a similar argument as in the case of AMALGAMATION above, IH gives, for each of the minor premises, a sequent $\Xi_i^* \leq \phi_i$ such that $S \vdash^- \Xi_i^* \leq \phi$ and Ξ_i is a b-version of Ξ_i^*. By the same sort of argument, there is a Γ^*, such that Γ is a b-version of Γ^* and $S \vdash^- \Gamma^*, \phi_0, \phi_1, ... \leq \phi$. Thus CUT in PLG suffices.
□

Theorem F.4 ***($\preceq$-Sufficiency)*** *Suppose σ is not a weak identity, but has the form $\phi \preceq \psi$, and let τ be a translation of σ. If $\mathbf{T}(S^{\prec}) \vdash^{s-} \tau$, then $S^{\prec} \vdash^- \sigma$.*

This result follows from a very simple induction on the depth of the derivation witnessing $\mathbf{T}(S^{\prec}) \vdash^{s-} \tau$.

Theorem F.5 ***($\prec$-Sufficiency)*** *Let σ be $\phi \prec \psi$. If, for all identification schemes f, $\mathbf{T}(f(S)) \vdash^{s-} f(\sigma)$, then $S \vdash^- \sigma$.*

Proof By D.7, it is enough to show that, if $(\forall f)\mathbf{T}(f(S^{\prec})) \vdash^{s-} f(\sigma)$, then $S \vdash \sigma$. Assume $\mathbf{T}(S^{\prec}) \vdash^{s-} \phi \prec \psi$. Let f be an identification scheme such that $f(\chi) = \phi$ if χ is a member of any purely weak chain of the form $\phi, ..., \psi$; and $f(\chi) = \chi$ otherwise. We'll show that if $\mathbf{T}(f(S^{\prec})) \vdash^{s-} f(\phi \prec \psi)$, then $S \vdash^- \phi \prec \psi$ by induction on the depth of the derivation d witnessing $T(f(S^{\prec}) \vdash^{s-} f(\phi \prec \psi)$. Note that theorem F.4 implies that $S \vdash^- \phi \preceq \psi$, so either $\phi = \psi$ or there is an S-chain of the form $\phi, ..., \psi$ by B.1. We may assume that any S-chain of the form $\phi, ..., \psi$ is purely weak, since otherwise theorem B.3 would give us $S \vdash^- \phi \prec \psi$ and we'd be done. Thus, $f(\phi) = f(\psi)$, and so $f(\phi \prec \psi) = \phi \prec f(\psi)$.

Basis case: Suppose $\phi \prec f(\psi) \in \mathbf{T}(f(S^{\prec}))$. $\phi \prec f(\psi) \in \mathbf{T}(f(S^{\prec}))$ iff there is a sentence χ such that $f(\chi) = f(\psi) = \phi$ and either $\phi \prec \chi \in S^{\prec}$ or $\phi \preceq \chi \in S^{\prec}$. But if $\phi \prec \psi \notin S^{\prec}$, $\phi \preceq \chi \in S^{\prec}$, and $f(\chi) = \phi$, then $f(\phi \preceq \chi)$ is a weak identity, and so $\phi \prec f(\psi) \notin T(f(S^{\prec}))$. So,

$$\phi \prec f(\psi) \in \mathbf{T}(f(S^{\prec})) \text{ iff } (\exists \chi)(f(\chi) = f(\psi) \text{ and } \phi \prec \chi \in S^{\prec})$$

If $f(\chi) = f(\psi) = \phi$, then there is a purely weak S-chain of the form $\phi, ..., \chi, ..., \psi$. Further, if $\phi \prec \chi \in S^{\prec}$, then there is a strict S-chain $\phi, ..., \chi$. So, there is a strict S-chain $\phi, ..., \psi$. Thus, theorem B.3 gives us $S \vdash^{-} \phi \prec \psi$.

Induction step: Suppose that $f(\phi \prec \psi)$ comes by TRANSITIVITY in LSG:

$$\frac{\phi \prec f(\chi) \quad f(\chi) \prec f(\psi)}{\phi \prec f(\psi)}$$

Suppose that $f(\chi) \neq \chi$. Then $f(\chi) = f(\psi)$, and IH suffices. So, suppose that $f(\chi) = \chi$. If $f(\chi) = \phi$, then IH suffices. Otherwise, it's easy to show that there is an S-chain $\phi, ...\chi$ and an S-chain $\chi, ..., \psi$. Thus, we are given an S-chain $\phi, ..., \chi, ...\psi$. Since $f(\chi) = \chi$, this last S-chain is not purely weak. Thus, there is a strict S-chain $\phi, ..., \psi$, and theorem B.3 yields the result.

Theorem F.6 ***(<-Sufficiency)*** *Let σ have the form $\Delta < \phi$. If, for every identification scheme f, there is a translation τ of $f(\sigma)$ such that $\mathbf{T}((f(S)) \vdash^{s-} \tau$, then $S \vdash^{-} \sigma$.*

Proof Suppose that $(\forall f)\mathbf{T}(f(S)) \vdash^{-}$ some translation of $f(\sigma)$. Now,

$$f(\Delta < \phi) = f(\delta_0), f(\delta_1), ... < f(\phi) \qquad (\delta_i \in \Delta).$$

So, for all identification schemes f and all members δ of Δ, $\mathbf{T}(f(S)) \vdash^{s-} f(\delta)^{\pm b} \prec f(\phi)$. If $\phi \in \Delta$, then any translation of $f(\Delta < \phi)$ has $f(\phi)$ (rather than just $b_{f(\phi)}$) on its left-hand side. Thus, by theorems D.4 and B.4, for all f and for all $\delta \in \Delta$, $\mathbf{T}(f(S)) \vdash^{s-} f(\delta) \prec f(\phi)$. Thus, theorem F.5 yields $S \vdash^{-} \delta \prec \psi$, for all $\delta \in \Delta$. An application of theorem F.3 yields $S \vdash^{-} \Delta \leq \phi$;[33] and so REVERSE SUBSUMPTION in PLG gives us $S \vdash^{-} \Delta < \phi$.
□

An immediate corollary of theorems F.3, F.4, F.5, and F.6 is that if S is consistent in PLG, then $S \vdash \sigma$ if $(\forall f)\mathbf{T}(f(S)) \vdash^{s-}$ some translation of $f(\sigma)$. This yields the following (qualified) conservativity result:

Theorem F.7 *If S is consistent in PLG, then $S \vdash \sigma$ iff for all identification schemes f, $f(\sigma)$ either is a weak identity or has a translation τ of $f(\sigma)$ such that $\mathbf{T}(f(S)) \vdash^{s-} \tau$.*

[33]if τ has the form $\Xi, \phi < \phi$, then it technically is not a translation of $\Xi, \phi \leq \phi$. But such a translation can easily be derived in LSG using the following instance of CUT.

$$\frac{b_\phi < \phi \quad \Xi, \phi < \phi}{\Xi . b_\phi < \phi}$$

Proof Suppose S is consistent in PLG. Theorem F.2 implies the biconditional in the left-to-right direction. Together, theorems F.3, F.4, F.5 and F.6 establish that if $(\forall f)\mathbf{T}(f(S)) \vdash^{s-}$ some translation of $f(\sigma)$, then $S \vdash^{-} \sigma$.
□

This allows the following characterization of derivability in PLG:

Theorem F.8 *$S \vdash \sigma$ iff either S is inconsistent in PLG, or, for all identification schemes f, if $f(\sigma)$ is not a weak identity, then it has a translation τ such that $\mathbf{T}(f(S)) \vdash^{s-} \tau$.*

Proof There are two cases:

(i) S is inconsistent in PLG; or

(ii) S is consistent in PLG.

In (i): For all sequents σ, $S \vdash \sigma$. Likewise, the first disjunct on the right-hand side of the theorem is true.
In (ii): Theorem F.7 yields the result.
□

And, of course, theorems F.1 and F.5 immediately imply that the consistency of S can be characterized in terms of derivability in LSG:

Theorem F.9 *S is inconsistent in PLG iff there is a sequent $\phi \prec \phi$ such that for all identification schemes f, $\mathbf{T}(f(S)) \vdash^{s-} f(\phi) \prec f(\phi)$.*

Thus, the derivability of a sequent from some premises in PLG can be exhaustively characterized in terms of derivability in LSG of a translation of the sequent from the translation of those premises.

The last theorem does not quite guarantee that when S is consistent there is an identification scheme under which the translation of S is also consistent. That would require there to be a single identification scheme f such that $\mathbf{T}(f(S))$ is consistent. However, it is easy to give a simple general recipe for constructing such an identification scheme given a consistent set of sequents.

Definition ϕ and ψ are *S-loopmates* iff either $\phi = \psi$ or there is a purely weak S-loop containing both ϕ and ψ.

It is obvious that *being S-loopmates* is an equivalence relation on the set of sentences. The equivalence class to which a given sentence ϕ belongs is given by:

$$\phi^\circ = \{\psi | \phi \text{ and } \psi \text{ are } S\text{-loopmates.}\}$$

Theorem F.10 *Let f_S° be an identification scheme such that*

$$(\forall\phi)(\forall\psi \in \phi^\circ)(\exists\chi \in \phi^\circ)f(\psi) = \chi.$$

If S is consistent, then $\mathbf{T}(f_S^\circ(S))$ is consistent.

Proof We first show the following lemma:

Lemma F.11 *If $\mathbf{T}(f_S^\circ(S^\prec)) \vdash^{s-} f_S^\circ(\phi) \prec f_S^\circ(\psi)$, then there is an $S^\prec$-chain of the form $\phi, ..., \psi$ which is not part of any purely weak $S^\prec$-loop.*

Proof The proof is by induction on the depth of the derivation witnessing $\mathbf{T}(f_S^\circ(S^\prec)) \vdash^{s-} f_S^\circ(\phi) \prec f_S^\circ(\psi)$. The induction step is trivial, so I'll only give the argument for the basis case.

Basis case: Suppose $f_S^\circ(\phi) \prec f_S^\circ(\psi) \in \mathbf{T}(f(S^\prec))$. Then either $\phi^* \prec \psi^*$ or $\phi^* \preceq \psi^*$ is in $S^\prec$, for $f_S^\circ(\phi) = \phi^*$ and $f_S^\circ(\psi) = \psi^*$. In the former case, ϕ^*, ψ^* is a strict $S^\prec$-chain, and so not part of any purely weak $S^\prec$-loop. In the latter case, $f_S^\circ(\phi^*) \neq f_S^\circ(\psi^*)$, since otherwise $f_S^\circ(\phi^* \preceq \psi^*)$ would be a weak identity. So, ϕ^* and ψ^* are not S-loopmates. Thus, ϕ^*, ψ^* is an S-chain and there is no purely weak S-loop of which ϕ^*, ψ^* is a part. If $\phi^* \neq \phi$, then there is an S-chain of the form $\phi, ..., \phi^*$. If $\psi \neq \psi^*$ then there is an S-chain of the form $\psi^*, ..., \psi$. Thus, there is an S-chain from ϕ to ψ, of which ϕ^*, ψ^* is a part. The result follows immediately.

□

Now, suppose that there is a sentence ϕ such that $\mathbf{T}(f_S^\circ(S)) \vdash^{s-} f_S^\circ(\phi) \prec f_S^\circ(\phi)$. By theorems D.7, D.1 and lemma F.11, there is an S-loop of the form $\phi, ..., \phi$ which is not part of any purely weak S-loop. Thus, the loop is not itself a purely weak S-loop.

□

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