The Tautochronous Pendulum.

Consider a cylinder of radius *a*. Let the cylinder roll to the right without slipping on a horizontal surface from initial point of contact O to new point of contact Q (see Figure 1). Let O be the <u>fixed</u> origin of the coordinate system. Note that the initial point of contact at O has moved to a new position P. The coordinates of that point are the cycloid coordinates:

$$x = a\theta - a\sin\theta \qquad (1)$$

$$y = a - a\cos\theta \qquad (2)$$

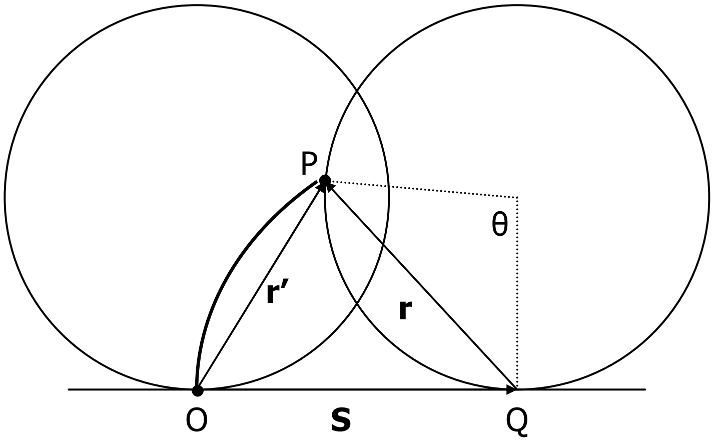

Figure 1.

Proposition I: Segment QP is perpendicular to the cycloid line at P.

<u>Proof</u>

We need show that the line element $d\mathbf{s}$ that is tangent to the cycloid at P is perpendicular to QP. Consider vectors **S**, **r** and **r'** as shown in the drawing. Clearly, **r** = **r'** – **S**. Then,

$\mathbf{r}' = (a\theta - a\sin\theta)\,\mathbf{e}_x + a(1-\cos\theta)\,\mathbf{e}_y$
$\mathbf{S} = a\theta\,\mathbf{e}_x$
$\mathbf{r} = \mathbf{r}' - \mathbf{S} = -a\sin\theta\,\mathbf{e}_x + a(1-\cos\theta)\,\mathbf{e}_y$

Now
$d\mathbf{s} = dx\,\mathbf{e}_x + dy\,\mathbf{e}_y = a(1-\cos\theta)\,d\theta\,\mathbf{e}_x + a\sin\theta\,d\theta\,\mathbf{e}_y$
and,
$d\mathbf{s}\cdot\mathbf{r} = -a^2\sin\theta(1-\cos\theta)\,d\theta + a^2\sin\theta(1-\cos\theta)\,d\theta = 0.$

Therefore, $d\mathbf{s}$ is perpendicular to QP at P, QED.

Proposition II: The length of arc segment OP along the cycloid is $s = 4a(1-\cos\frac{\theta}{2})$

<u>Proof</u>

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$dx = (1-\cos\theta)a\,d\theta; \quad dy = a\sin\theta\,d\theta$$

$$ds = a\,d\theta\sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = a\,d\theta\sqrt{2(1-\cos\theta)} = 2a\sin\tfrac{\theta}{2}\,d\theta$$

$$s = 2a\int_0^\theta \sin\tfrac{\theta}{2}\,d\theta = 4a\Big[-\cos\tfrac{\theta}{2}\Big]_0^\theta = 4a\Big(1-\cos\tfrac{\theta}{2}\Big)$$

QED.

Proposition III: Consider two identical cylinders, bottom and top. They roll without slipping on bottom and top surfaces S_B and S_T with identical velocities to the right (see figure 2). Let O_B and O_T be cycloid generator points for each cylinder. **When the cylinders have rotated by angle θ, line segment P_TP_B is tangent to the cycloid at point P_B.**

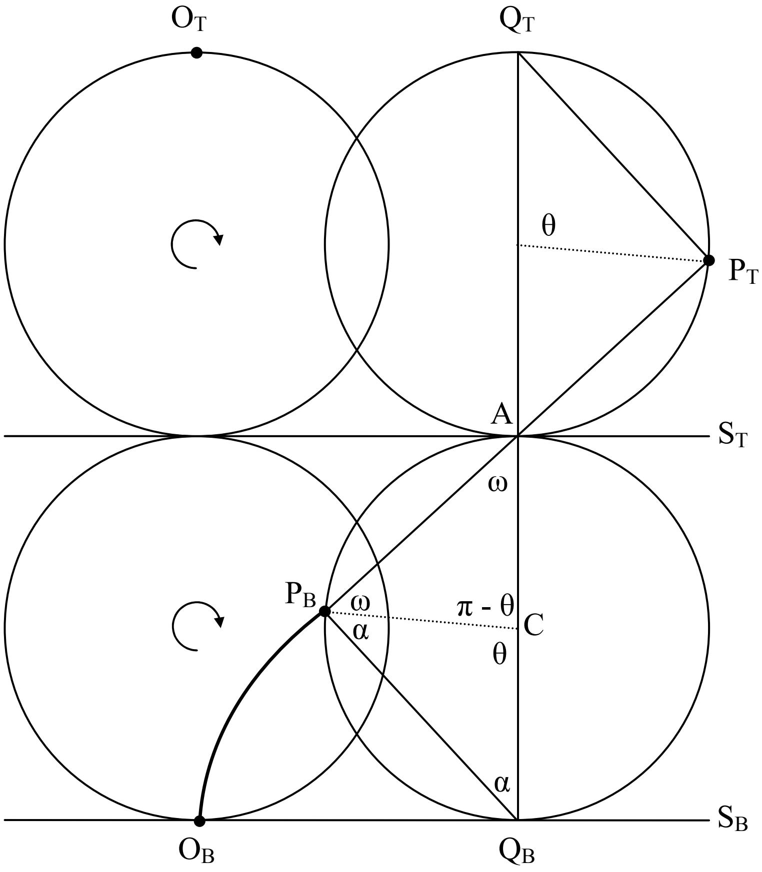

Figure 2.

Proof:

We need to show that segment P_TP_B is perpendicular to segment Q_BP_B at point P_B. Then, by Proposition I, segment P_TP_B is tangent to the cycloid at point P_B.

First we note that, by symmetry, triangles AQ_BP_B and AQ_TP_T are congruent. This means that line P_TP_B has to pass through point A, the point at which the cylinders are tangent to one another.

From isosceles triangle $\triangle C P_TP_B$ we have

$$2\alpha + \theta = \pi \quad \rightarrow \quad \alpha = \frac{\pi}{2} - \frac{\theta}{2}$$

and from isosceles $\triangle CAP_B$

$$2\omega + (\pi - \theta) = \pi \quad \rightarrow \quad \omega = \frac{\theta}{2}$$

Adding the two equations gives $\alpha + \omega = \frac{\pi}{2}$ which proves the proposition. It follows from the above that segment P_TP_B is perpendicular to the tangent of the top cycloid at point P_T.

Proposition IV: The length of curvilinear segment $O_BP_BP_T$ is constant and equal to 4*a*.

To find the length of the curvilinear segment, we need to add the curved segment *s* from Proposition II, to the length of straight line segment P_BP_T. The coordinates of P_B are given by equations (1) and (2):

$$x_B = a\theta - a\sin\theta$$
$$y_B = a - a\cos\theta$$

The equation fro the top cycloid with origin at O_B can easily be found to be

$$x_T = a\theta + a\sin\theta$$
$$y_T = 3a + a\cos\theta$$

The length of segment P_BP_T is

$$d = \sqrt{(x_T - x_B)^2 + (y_T - y_B)^2} = \sqrt{(2a\sin\theta)^2 + (2a + 2a\cos\theta)^2} = 2a\sqrt{2(1+\cos\theta)}\,.$$

Now,

$$1 + \cos\theta = 1 + \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} = 2\cos^2\tfrac{\theta}{2}\,.$$

Therefore,

$$d = 4a\cos\tfrac{\theta}{2}$$

The length of the curvilinear segment is the sum:

$L = s + d = 4a\left(1 - \cos\tfrac{\theta}{2}\right) + 4a\cos\tfrac{\theta}{2} = 4a$, QED.

Now pretend that gravity acts (unconventionally) from bottom to top. If a string of length 4*a* is attached to a fixed support at O_B and a mass is attached to the other end at point P_T, the mass will move under the influence of gravity in a cycloid, i.e. we have the brachistochrone problem with the following non-essential difference: In the brachistochrone problem the track provides the normal force; here it is the tension that provides the normal force.

In the breachistochrone, we have seen that the time required for the mass to reach the horizontal position from <u>any</u> starting position is the same and equal to $\pi\sqrt{a/g}$. This says that the tautochronous pendulum has a period that is independent of the amplitude. Since the time required to reach the equilibrium position starting from anywhere is one-quarter of the period, the period of the tautochronous pendulum is $T = 4\pi\sqrt{a/g}$.