Computing the Canonical Ring of Certain Stacks

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Abstract

We compute the canonical ring of some stacks. We first give a detailed account of what this problem means including several proofs of a famous historical example. The main body of work of this thesis expands on our article [Fra23] in describing the geometry of Drinfeld modular forms as sections of a specified line bundle on a certain stacky modular curve. As a consequence of that geometry, we provide a program: one can compute the (log) canonical ring of a stacky curve to determine generators and relations for an algebra of Drinfeld modular forms, answering a problem posed by Gekeler in 1986.

Et si tu crois que je m'en fous
Que l'amour nous a mis à bout
J'ai encore des larmes de réserves
J'ai encore des drames
que j'préserve

Et si tu crois que je m'en fous
Que l'amour nous a mis à bout
J'ai pas vu passer nos amours
J'ai pas vu passer le jour

"Le Jour" - Al'Tarba, Mounika

Fifth grade, smarter than my parents
Grandma couldn't help with algebra
Grandma like,"What the fuck is algebra?"
She like, "That's a goddamn shame
them people gon' keep makin' up shit
tryna keep you in the same grade."
They tryna hide shit in the book

From "American Tterroristt" - RXKNephew

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Chapter 1

Overview

1.1 History

The theory of modular forms in the classical number-field case has existed since the 1800's. It is well-understood that modular forms are sections of a particular line bundle on some stacky modular curve. In this set up the geometry of the stacks, with tools such as the Riemann-Roch theorem for stacky curves for example, can be used to compute section rings which describe algebras of modular forms. The program of [VZB22] for computing the canonical ring of log stacky curves in all genera even gives minimal presentations for many such section rings, that is: explicit generators and relations, which correspond to generators and relations for algebras of modular forms.

Drinfeld introduced the study of what he called "elliptic modules," which we now call "Drinfeld modules" in 1974 with [Dri74] in order to address problems in the Langlands program over function fields. Many objects from classical number theory such as modular curves and modular forms have analogs over function fields, and we refer to the function-field side of this analogy as the "Drinfeld setting."

In his 1986 monograph [Gek86, Page XIII] asks for a description of algebras of Drinfeld modular forms in terms of generators and relations. The main results of this thesis describe the geometry of those modular forms, which allows one to employ techniques such as those in [VZB22] to find the desired generators and relations by considering the geometry of the corresponding Drinfeld modular curve. That is, we provide a means to address Gekeler's problem via geometric invariants.

There is a collection of results which is similar to our work in comparing modular forms for various congruence subgroups to each other as in our second main result Theorem 7.2.1. Pink finds isomorphisms between algebras of Drinfeld modular forms for open compact subgroups $K \leqslant \mathrm{GL}_r(\widehat{\mathbb{F}_q[T]})$, where the hat symbol denotes the pro-finite completion $\widehat{\mathbb{F}_q[T]} = \prod_p (\mathbb{F}_q[T])_p$, and normal subgroups $K' \lhd K$ in e.g. [Pin12, Proposition 5.5]. Pink also describes Drinfeld modular forms as sections of an invertible sheaf in [Pin12, Section 5] which is similar to Theorem 7.1.1. However, Pink needs the dual of the relative Lie algebra over a line bundle, rather than the bundle itself, to describe Drinfeld modular forms, which is a major difference with our work.

There are some existing results which approach Gekeler's problem, such as Cornelissen's papers [Cor97a] and [Cor97b] which handle linear level in [Cor97b, Theorem (3.3)], i.e. the algebra of modular forms for $\Gamma(\alpha T + \beta)$, where $\alpha \in \mathbb{F}_q^\times$ and $\beta \in \mathbb{F}_q$, and include some results for quadratic level in [Cor97b, Proposition (3.4)]. Another example, [DK23, Theorem (4.4)], computes the algebra of Drinfeld modular forms for $\Gamma_0(T)$. The best known result for general level N is from e.g. [Arm08, Proposition 4.16] which demonstrates that for any level the double cusp forms of weight 2 and type 1, which form the vector space $M^2_{2,1}(\Gamma_0(N))$, are (analytic) holomorphic differentials on a (rigid analytic) Drinfeld modular curve $\Gamma_0(N)\backslash(\Omega \cup \mathbb{P}^1(\mathbb{F}_q(T)))$, where Ω is the Drinfeld "upper half-plane" defined in Section 4.1.

Several ideas in [Bre16] are central to our argument, as well as being an exposition on aspects of Gekeler's problem in general. In particular, [Bre16] introduces the subgroup Γ_2 of a given congruence subgroup $\Gamma \leqslant \mathrm{GL}_2(\mathbb{F}_q[T])$ and gives a moduli interpretation of the corresponding Drinfeld modular curve.

Even by the date of these most recent papers, the generalization to the algebra of modular forms for $\Gamma_0(N)$ for any level N, all examples of modular forms for $\Gamma_1(N)$, and higher level (i.e. $\deg(N) \geqslant 2$) examples for $\Gamma(N)$ seem to be wide open. Similarly, other than some preliminary results such as formulae for geometric invariants in [GvdP80] it is also an open problem to compute generators and relations for algebras of Drinfeld modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{F}_q[T])$.

Our work differs considerably from the papers of Armana, Breuer, Cornelissen, Dalal-Kumar, and Gerritzen-van der Put cited above in that we work with Drinfeld moduli stacks as opposed to schemes. As early as [Gek86] and [Lau96] it was known that moduli of Drinfeld modules of fixed rank are Deligne-Mumford stacks, but it is the more recent results of [VZB22] for computing log canonical rings of stacky curves, and [PY16] which provides a crucial principle of rigid analytic GAGA (short for "géométrie algébrique et géométrie analytique") for stacks, that makes our work possible.

There is some historical reason to work with rigid analytic spaces as opposed to the more general adic or Berkovich spaces, namely the original analytic theory of the Drinfeld setting was developed in that language in e.g. Goss's paper [Gos80]. Though there is for example a more general or modern theory of adic stacks (see e.g. [War17]) we will find it more convenient to phrase things in terms of rigid analytic spaces, and there is no loss in doing so.

1.2 Organization of this Work

In Chapter 2 we define the canonical ring of a scheme. This discussion is an elementary introduction to our theory in general in the sense that we make several arguments from first principles, carefully define many fundamental objects, and repeat some famous historical calculations.

Appendix A motivates the calculation of canonical rings. After introducing some terminology and a brief interlude about the infinitesimal lifing property, we discuss the construction of Proj of a graded ring and define morphisms of schemes. The point of this material is to be able to prove that with some standard simplifying assumptions, as a scheme a curve is isomorphic to its image under the canonical embedding, and this embedded curve is Proj of the canonical ring.

Appendix B gives yet another proof of Petri's Theorem distinct from the two we discuss in Chapter 2. The idea with this version is to discuss Green and Lazarsfeld's "simple proof" of Petri [GL85] which is purely cohomological. We comment on why such a technique is interesting in the introduction to this Appendix, and then give a highly detailed account of Green and Lazarsfeld's proof.

We define stacks in Chapter 3, which for experts is our true starting place. The first part of this discussion is a development of the notion that stacks are a 2-categorical version of a sheaf. We hope this introduces stacks by analogy with sheaves which are more familiar, but in later Chapters we use a more practical working definition of a stack that we state after the analogy. We also define the specific invariants of a stacky curve that we use to compute (log) canonical rings, and comment on existing results in this direction.

Chapter 4 introduces the Drinfeld, or function-field setting. We focus on describing the analogy between function fields and number fields, the latter being the so-called classical setting for arithmetic geometry. We also describe Drinfeld modules, which are a version of abelian varieties in this setting. Rather than work in the greatest generality possible over the function field of any smooth, projective, connected curve over some field of positive characteristic, we content ourselves with working with the the function field of the curve $\mathbb{P}^1$. This makes the polynomial ring $\mathbb{F}_q[T]$, for q a power of an odd prime and T an indeterminant, our "integers" so that we can simplify the discussion somewhat.

Our work begins in earnest in Chapter 5. We need not only a version of the classic GAGA theory for rigid analytic spaces, but also one for rigid analytic stacks. As such, we recall the theory of rigid analytic spaces and rigid GAGA, then define rigid analytic stacks, and finally we state the main GAGA results we use.

Chapter 6 is an introduction to Drinfeld modular curves and Drinfeld modular forms. These are the main objects of study for this work. We discuss a (Satake) compactification of Drinfeld moduli and some local (rigid) analysis near the points added in this compactification. We also give moduli interpretations for some stacky Drinfeld modular curves.

Our main results about the geometry of Drinfeld modular forms are in Chapter 7. We find a Drinfeld modular curve and a specific line bundle on that curve whose sections are Drinfeld modular forms for congruence subgroups Γ containing the diagonal matrices in $\mathrm{GL}_2(\mathbb{F}_q[T])$ and such that $\det(\gamma) \in (\mathbb{F}_q^\times)^2$ for every $\gamma \in \Gamma$ in Theorem 7.1.1. This gives us a way to answer Gekeler's problem for modular forms for Γ so long as Γ satisfies our hypotheses. Then we show how the algebra of Drinfeld modular forms for some congruence subgroup Γ' can be expressed in terms of a direct sum of components of the algebra of Drinfeld modular forms for another congruence subgroup Γ which contains Γ'. This comparison of algebras means we can compute generators and relations for algebras of modular forms for congruence subgroups which may not contain only square-determinant matrices. We illustrate this theory with a special case: Theorem 7.2.1. We generalize Theorem 7.2.1 with Theorem 7.3.1 which compares algebras of modular forms for Γ and some of its subgroups Γ', generalizing the special case $\Gamma' = \Gamma_2$ from Theorem 7.2.1.

Finally, in Chapter 8 we give an algorithm which solves Gekeler's problem for certain congruence subgroups, up to the user's being able to compute the log canonical ring of a given stacky curve. We give some examples of our application of this technique to repeat known results. We conclude with some comments on several cases which should be tractable and very interesting to consider in future work.

1.3 Main Results

Let Γ be a congruence subgroup of $\mathrm{GL}_2(\mathbb{F}_q[T])$. Suppose that Γ contains the scalar matrices of $\mathrm{GL}_2(\mathbb{F}_q[T])$ and $\det(\gamma) \in (\mathbb{F}_q^\times)^2$ for every $\gamma \in \Gamma$. First, we show that the Drinfeld modular forms for such Γ are sections of a log canonical bundle on the associated stacky Drinfeld modular curve $\mathscr{X}_\Gamma$. Note that this solves Gekeler's problem for groups satisfying our hypotheses, assuming we can compute the generators and relations of the log canonical ring of the stacky curve.

Theorem 1.3.1 (Theorem 7.1.1 in the text)**.** *Let q be an odd prime and let $\Gamma \leqslant \mathrm{GL}_2(\mathbb{F}_q[T])$ be a congruence subgroup containing the scalar matrices of $\mathrm{GL}_2(\mathbb{F}_q[T])$ and such that $\det(\gamma) \in (\mathbb{F}_q^\times)^2$ for every $\gamma \in \Gamma$. Let Δ be the divisor supported at the cusps of the modular curve $\mathscr{X}_\Gamma$ with rigid analytic coarse space $X_\Gamma^{an} = \Gamma\backslash(\Omega \cup$*

$\mathbb{P}^1(\mathbb{F}_q(T)))$. *There is an isomorphism of graded rings*

$$M(\Gamma) \cong R(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(2\Delta)),$$

where $\Omega^1_{\mathscr{X}_\Gamma}$ *is the sheaf of differentials on* $\mathscr{X}_\Gamma$. *The isomorphism of algebras is given by the isomorphisms of components* $M_{k,l}(\Gamma) \to H^0(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(2\Delta)^{\otimes k/2})$ *given by* $f \mapsto f(dz)^{\otimes k/2}$.

To handle the more general case of congruence subgroup Γ which contains the diagonal matrices of $\mathrm{GL}_2(\mathbb{F}_q[T])$ but which may not contain only square-determinant matrices, we consider the normal subgroup $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$ of Γ. We compare the algebras of Drinfeld modular forms for Γ and Γ_2 and arrive at the following result. Note that this reduces giving an answer to Gekeler for the congruence subgroups Γ to computing log canonical rings of stacky Drinfeld modular curves.

Theorem 1.3.2 (Theorem 7.2.1 in the text)**.** *Let q be a power of an odd prime. Let* $\Gamma \leqslant \mathrm{GL}_2(\mathbb{F}_q[T])$ *be a congruence subgroup containing the diagonal matrices in* $\mathrm{GL}_2(\mathbb{F}_q[T])$. *Let* $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$. *Then as rings* $M(\Gamma) \cong M(\Gamma_2)$, *with*

$$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$

on each graded piece, where l_1, l_2 *are the two solutions to* $k \equiv 2l \pmod{q-1}$.

Finally, we generalize the previous comparison theorem to a larger class of subgroups $\Gamma' \leqslant \Gamma$, where Γ is some chosen or distinguished congruence subgroup as above. This idea was proposed in correspondence by Gebhard Böckle, as was the proof technique which we execute. This result is similar to classical results about nebentypes of modular forms.

Theorem 1.3.3 (Theorem 7.3.1 in the text)**.** *Let q be a power of an odd prime. Let $\Gamma \leqslant \mathrm{GL}_2(\mathbb{F}_q[T])$ be a congruence subgroup. Let $\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$. Suppose that Γ' is such that $\Gamma_1 \leqslant \Gamma' \leqslant \Gamma$. Then as algebras*

$$M(\Gamma) = M(\Gamma'),$$

and each component $M_{k,l}(\Gamma')$ is some direct sum of components $M_{k,l'}(\Gamma)$ for some nontrivial l'.

Chapter 2

What are Canonical Rings?

To introduce the theory of canonical rings we consider the following version of Petri's theorem. Let X be a genus $g \geqslant 4$ canonical (non-hyperelliptic), smooth, irreducible, projective, complex algebraic curve and let ω_X denote the canonical bundle on X. The assumption that we work over the complex numbers is purely for convenience as we discuss in Remark 2.0.1. As we will see, ω_X defines a closed immersion $\varphi : X \to \mathbb{P}^{g-1}$. Let $R = R(X, \omega_X) = \oplus_{d \geqslant 0} H^0(X, \omega_X^{\otimes d})$ denote the canonical ring of X in $\mathbb{P}^{g-1}$. Then Petri's theorem says $R \cong \mathbb{C}[x_1, \cdots, x_g]/I$, where R is generated in degree 1, the ideal I is generated in degree 2 (by 'quadrics') and when $g = 5$ in particular, $\dim_{\mathbb{C}} I_2 = 3$. That is, informally, genus 5 curves are the complete intersection of 3 quadrics in $\mathbb{P}^4$.

The full statement of Petri's theorem relates the geometry of a curve with genus $g \geqslant 4$ to the structure of its canonical ring $R_C = R(C, \omega_C)$ and concludes that R is generated in degree 1 with relations in degree 2 unless C is hyperelliptic, trigonal or a plane quintic (see e.g. [ACGH85, Section 3.3]). We often focus on the case of genus $g = 5$ where we obtain a particularly nice description of the canonical ring, and can illustrate many calculations explicitly while keeping the notation somewhat readable.

In this chapter we will discuss both a genus formula for complete intersections and directly consider the ideal of relations for a genus $g \geqslant 4$ curve which is canonically embedded into $\mathbb{P}^{g-1}$. Along the way we define many fundamental objects such as curves and their canonical bundles, so that while the document is not entirely self-contained, it at least proceeds from a reasonably elementary point.

Remark 2.0.1. *Throughout this chapter we work over an algebraically closed field* $\mathbb{F} = \overline{\mathbb{F}}$ *with* $\mathrm{char}(\mathbb{F}) = 0$, *typically* $\mathbb{C}$. *As in [LRZ18b, Remark* 2.1.1*], the assumption of algebraic closure is not essential at all, but merely for convenience. The graded pieces of the canonical ring are preserved under base change from* $\mathbb{F}$ *to* $\overline{\mathbb{F}}$ *since flat base-change commutes with cohomology. Indeed, even though over an inseparable extension of the base field, the base change of the canonical bundle may not agree with the canonical bundle of the base change, the structure of a canonical ring does not change when base changing from* $\mathbb{F}$ *to its algebraic closure* $\overline{\mathbb{F}}$. *That is, generators and relations for a canonical ring are preserved under base field extension, as are their minimal degrees.*

2.1 Notation and Preliminaries

In this section, we will define a topology which we use throughout this chapter, and state one computational Theorem. These facts are found in standard treatments such as [Har77].

For expert readers, we begin by specifying a Grothendieck topology for our schemes, and for the non-expert, we define a topology that we will use on our schemes.

Definition 2.1.1 ([Har77, pages 9 – 10])**.** *Let S be a graded ring, let $f \in S$ be a homogeneous polynomial, and denote by S_+ the maximal ideal $S_+ = \oplus_{d>0} S_d$. We define*

$$V_+(f) \overset{def}{=} \{p \trianglelefteq S : p \textit{ is a homogeneous prime ideal, } p \neq S_+, \textit{ and } f = 0 \pmod{p}\},$$

and if $a \trianglelefteq S$ is any homogeneous ideal, we define the ***zero set of*** *a:*

$$Z(a) \overset{def}{=} Z(T) = \{p \in S : f(p) = 0 \textit{ for all } f \in \{\textit{homogeneous elements of } a\}\},$$

where T is the set of all homogeneous elements of a. Finally, we say that a subset $Y \subset \mathbb{P}^n_S$ is an ***algebraic set*** *if there exists a set T of homogeneous elements such that $Y = Z(T)$ and define the* ***Zariski topology*** *on $\mathbb{P}^n_S$ by taking open sets to be the complements of algebraic sets.*

Example 2.1.2. *The Zariski topology on $\mathbb{P}^N_{\mathbb{C}}$ has a basis of the open sets of the form $D_+(f)$, the nonvanishing locus of the function $f \in \mathbb{C}[x_0, \cdots, x_N]$ as f varies.*

By means of defining as little as possible to get as much done as we can, we state only a few things which appear in the detailed anatomy of a sheaf on a scheme.

Definition 2.1.3. *Let X be a scheme over $\mathbb{C}$ and let $\mathcal{F}$ be a sheaf on X. Then for an open $U \subseteq X$, the elements $s \in \mathcal{F}(U)$ are called the* ***sections*** *of $\mathcal{F}$, and in particular are called* ***global sections*** *when $U = X$. Write $H^0(X, \mathcal{F})$ for the $\mathbb{C}$-vector space of* *global sections of $\mathcal{F}$, and let*

$$h^0(X,\mathcal{F}) \overset{def}{=} \dim_{\mathbb{C}} H^0(X,\mathcal{F}).$$

We need a notion of functions on our scheme for the theory which follows. In scheme-theoretic terminology, this means defining a sheaf of rings (of functions).

Definition 2.1.4 ([Har77, page 110])**.** *Let A be a ring, let $p \trianglelefteq A$ be a prime ideal and denote the localization of A at p by A_p. Suppose X is scheme over* $\operatorname{Spec} A$. *The* ***structure sheaf*** *$\mathcal{O}_X$ on X is the sheaf of rings defined on each open $U \subseteq X$ to be the ring of functions $s : U \to \bigsqcup_{p\in U} A_p$ such that for each $p \in U$, $s(p) \in A_p$ and for each $p' \in U$ there is some open neighborhood V of p' contained in U and elements $a, f \in A$ such that for each $q \in V$, $f \notin q$ and $s(q) = a/f$ in A_q.*

Finally, turning to the sheaves of modules which appear later in the document, we introduce one last purely sheaf-theoretic idea.

Definition 2.1.5. *Let X be a scheme over $\mathbb{C}$. For E a locally free sheaf of rank r on X, the* ***determinant*** *of E is* $\det(E) \overset{def}{=} \bigwedge^r E$, *where $\bigwedge^r$ denotes the rth exterior product, i.e. the rth graded component of the exterior algebra.*

Later we will want to compute the determinant of a vector bundle, i.e. a sheaf such as E in Definition 2.1.5, which we can do via the following theorem:

Theorem 2.1.6. *Let X be a scheme over $\mathbb{C}$ and let E, F, G be locally free sheaves on X. If $0 \to E \to F \to G \to 0$ is exact then*

$$\det(E) \otimes \det(G) \cong \det(F).$$

Proof. This is [Har77, Exercise II.6.11]. □

2.2 Curves and Complete Intersections

The true starting point for the theory which we cover in this thesis is the definition of a curve. We specify what kind of curve we consider by in particular the notions of a "projective" scheme, an assumption we make about curves, and "complete intersections" which we discuss in great detail. These choices mean we do not have to make a choice about some kind of ambient space in which our curve could live, but rather let us inherit the geometry of a well-known space in our consideration of some subspace, and that we use the theory of intersections in algebraic geometry, respectively.

Definition 2.2.1. *A* ***curve*** *is an integral, smooth, projective, Noetherian, separated, one-dimensional scheme of finite type over* $\mathrm{Spec}(\mathbb{F})$ *for* $\mathbb{F}$ *some (algebraically closed) field.*

In particular, by projective, we mean X is an irreducible algebraic set in $\mathbb{P}^N$ with the induced subset topology. With this notion, we can begin to introduce ring-theoretic objects associated to a curve.

Definition 2.2.2 ([Har77, page 10])**.** *Suppose* X *is any subset of* $\mathbb{P}^N_{\mathbb{F}}$ *where* $\mathbb{F}$ *is an algebraically closed field. The* ***homogeneous ideal*** *of* X*, denoted* $I(X)$*, is the ideal generated by*

$$\{f \in \mathbb{F}[x_0, \cdots, x_N] : f \textit{ is homogeneous and } f(P) = 0 \textit{ for all } P \in X\}.$$

Next, by using the ideal above, we begin a special case of intersection theory.

Definition 2.2.3 ([Har77, Exercise I.2.17])**.** *A variety $X \subset \mathbb{P}^N$ of dimension n is a (strict)* ***complete intersection*** *if $I(X)$ can be generated by $N-n$ elements. We say X is a* ***set-theoretic complete intersection*** *if X can be written as the intersection of $N-n$ hypersurfaces.*

The next result allows us to compute the degree of a complete intersection of hypersurfaces of known degrees.

Theorem 2.2.4 ([EH16, Corollary 1.24])**.** *If c hypersurfaces $Z_1, \cdots, Z_c \subset \mathbb{P}^N$ meet in a scheme X of codimension c with irreducible components $C_1, \cdots, C_t$ then*

$$\sum \deg[C_i] = \prod \deg[Z_i].$$

Corollary 2.2.5. *The degree of a complete intersection of hypersurfaces D_1, D_2 and $D_3 \subset \mathbb{P}^N$ of degrees d_1, d_2 and d_3 respectively, which intersect in a curve, is $d_1d_2d_3$.*

Proof. Since a curve is 1-dimensional, it has codimension N in $\mathbb{P}^N$. Since a smooth curve is irreducible, i.e. has a unique component, the degree of the complete intersection of hypersurfaces which meet in a smooth curve is the product of their degrees. $\square$

2.3 Bundles

In this section we consider certain sheaves of free modules on a scheme, which we call vector bundles. This discussion is separated into a discussion of features of, computational tools for, constructions of, and then examples of different bundles. These topics meet in the discussion of the canonical bundle, and the canonical ring in particular, for a curve. While there are probably treatments of each of the facts in this section in [Har77], we cite a variey of sources instead, both for readability as well as proximity to the overarching problem of the genus formula for complete intersections that we aim to state.

Formally, we consider the following kinds of sheaves of modules throughout the rest of these notes.

Definition 2.3.1. *Let X be a curve over $\mathbb{F}$. Then a **vector bundle of rank** n on X is a locally free sheaf of rank n $\mathcal{O}_X$-modules. A **line bundle** on X is a vector bundle of rank* 1.

To make the abstract notion of a sheaf such as a line bundle more convenient for computation, we will often use the following notion of divisors in place of line bundles. Indeed in many situations, such as the case of a smooth curve, there is a correspondence between line bundles and divisors. When working with line bundles on a curve which is not smooth, only certain special divisors called Cartier divisors correspond to line bundles. First we introduce the notion of divisors.

Definition 2.3.2. *Let X be a scheme of dimension n over an algebraically closed field $\mathbb{F}$. Then a **divisor** on X is a formal sum of codimension* 1 *subschemes of X.*

We have a certain uniqueness condition for divisors from the following notion of linear equivalence between them.

Definition 2.3.3. *Let X be a scheme of dimension n over an algebraically closed field $\mathbb{F}$ with function field $\kappa(X)$. We say that two divisors D and E on X are **linearly equivalent** if there is some $f \in \kappa(X)$ such that $\operatorname{div}(f) \overset{def}{=} Z(f)-P(f) = D-E$, where*

$Z(f)$ and $P(f)$ respectively denote the zeros and poles of f, counting multiplicities. Write $\mathrm{Div}(X)$ *for the free abelian group of divisors up to linear equivalence on X.*

Example 2.3.4. *When $X \subset \mathbb{P}^N$ is a curve, a divisor on X is a formal sum of points on X.*

Remark 2.3.5. *Since in these notes we consider the particular case when X is a smooth curve, we will conflate line bundles and divisors on X. When the hypothesis of smoothness is relevant, we will denote the missing assumption that the corresponding divisors in question are Cartier with parenthesis.*

We can naively spell out the correspondence between line bundles and divisors quite neatly. Given $\mathcal{L}$ a line bundle on an integral scheme X and s a rational section of $\mathcal{L}$, the associated divisor is

$$\mathrm{div}(s) = Z(s) - P(s) \in \mathrm{Div}(X).$$

Conversely, given $D = \sum n_i P_i$ a (Cartier) divisor on X, the sheaf $\mathcal{O}_X(D)$ is a line bundle on X, where

$$\mathcal{O}_X(D) = \{f \in \kappa(X) : f \text{ has a poles at worst } D\},$$

and $\kappa(X)$ is the function field of X.

2.3.1 Facts about Bundles

Line bundles can define rational maps to projective space.

Definition 2.3.6. *Let X be a scheme over an algebraically closed field $\mathbb{F}$ and let $\mathcal{L}$ be a line bundle on X. Suppose $s_0, \ldots, s_r$ is a basis for $H^0(X, \mathcal{L})$. Then there exists a rational map*

$$\varphi_{\mathcal{L}} : X - \{s_0 = \cdots = s_r = 0\} \to \mathbb{P}^r$$

given by $P \mapsto (s_i(P))_{i=0}^r$.

Remark 2.3.7. *This is a rational map in the sense that it is defined only on a dense open subset of X rather than the full space. In particular if $\mathcal{L}$ is a basepoint-free line bundle, i.e. $\{s_0 = \cdots = s_n = 0\} = \varnothing$, then $\varphi_{\mathcal{L}}$ may define a map to $\mathbb{P}^r$ on all of X.*

We use the following terminology to describe whether the induced maps from line bundles somehow preserve the geometry of the scheme they are defined on.

Definition 2.3.8 ([Sta18a]). *Say a line bundle $\mathcal{L}$ is **very ample** if the map $\varphi_{\mathcal{L}} : X \to \mathbb{P}^r$ by global sections of $\mathcal{L}$ is a closed immersion as in [Sta18b, Tag 01QN]. Say the line bundle $\mathcal{L}$ is **ample** if there is some nonnegative $r \in \mathbb{Z}$ such that $\mathcal{L}^{\otimes r}$ is very ample.*

Next, we do an apriori ring-theoretic construction on sheaves of modules over Proj of a graded ring. This is an extended example of a line bundle which not only lies on the ambient projective scheme which our (embedded) curves live in, but also deals explicitly with hypersurfaces, which we will see cut out our curves as complete intersections.

We start our construction with a fact about graded rings.

Definition 2.3.9. *Let $S = \oplus_{e \geqslant 0} S_e$ be a graded ring. The d**th Serre twist** of S is the S-module $S(d)$ given by $S(d)_e \overset{def}{=} S_{e+d}$.*

Now we introduce a sheaf on Proj of a graded ring.

Definition 2.3.10. *Let S be a graded ring and let M be a graded S-module. Then there is a sheaf of modules $\tilde{M}$ on* $\operatorname{Proj}(S)$ *defined by (the sheafification of)*

$$\tilde{M}(D_+(f)) = M\left[\frac{1}{f}\right]_0,$$

where $M[\frac{1}{f}]_0$ is the 0th graded component of $M[\frac{1}{f}]$.

Finally, we relate this sheaf of modules to the structure sheaf.

Definition 2.3.11. *Let S be a graded ring and write $\mathbb{P}^N_S$ for* $\operatorname{Proj} S[x_0, \cdots, x_N]$ *for x_i indeterminates. The* ***Serre twisting sheaf*** *$\mathcal{O}_{\mathbb{P}^N_S}$ on $\mathbb{P}^N_S$, is $\mathcal{O}_{\mathbb{P}^N_S}(d) \overset{def}{=} \widetilde{S_{\mathbb{P}^N_S}(d)}$.*

The following Theorem is instrumental computing the Picard group of $\mathbb{P}^N$ as well as making tensor products of line bundles on $\mathbb{P}^N$ into a problem about elementary addition of degrees.

Theorem 2.3.12. *Let $\mathbb{F}$ be a field. For any $d \in \mathbb{Z}_{\geqslant 0}$*

$$\mathcal{O}_{\mathbb{P}^N_{\mathbb{F}}}(d) \cong \mathcal{O}_{\mathbb{P}^N_{\mathbb{F}}}(dH),$$

for $H \subset \mathbb{P}^N_{\mathbb{F}}$ any hyperplane.

Proof. Let $S = \mathbb{F}[x_0, \cdots, x_N]$ and fix d a non-negative integer. Recall that $S(d) \overset{def}{=} S_{e+d}$ by Definition 2.3.9 so $S(d) = \oplus_{e\geqslant 0}\mathbb{F}[x_0, \cdots, x_N]_{e+d}$. Let $\widetilde{S}$ be the sheaf of S-modules on $\operatorname{Proj} S \cong \mathbb{P}^N_{\mathbb{F}}$ given by (sheafifying)

$$\widetilde{S(d)}(D_+(f)) \cong S\left[\frac{1}{f}\right]_d = \mathbb{F}\left[x_0, \cdots, x_N, \frac{1}{f}\right]_d,$$

where the nontrivial isomorphism of localized rings is from Example 2 on page 708 in [DF04]. Fix an affine open cover

$$\mathbb{P}^N_{\mathbb{F}} = \bigcup_{i=0}^{N} U_i = \bigcup_{i=0}^{N} D_+(x_i),$$

and for each affine open U_i where $0 \leqslant i \leqslant N$ consider a map $\varphi_i : \widetilde{S(d)}(U_i) \to \mathcal{O}_{\mathbb{P}^N_{\mathbb{F}}}(dH)(U_i)$ given by

$$f \mapsto f(\pi(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N))$$

for $f \in \mathbb{F}[x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]_d$, where $\pi \in S_{N-1}$ is a permutation of indices of coordinates. Note that we might equivalently define our map by a composition of f with a linear change of basis for homogeneous degree d polynomials in $x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N$. In other words, each φ_i is a composition of the identity map on $\mathbb{F}[x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]_d$ with an automorphism of $\mathbb{F}[x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]_0$, and therefore is a well-defined ring homomorphism. There is a well-defined injective inverse map by composing f^{-1} with the inverse permutation-of-coordinates or respectively the inverse of the change-of-basis automorphism, i.e. $\pi^{-1} \circ f^{-1}$, and therefore on each affine open we have an isomorphism. This way we have isomorphisms

$$\varphi_{ij} \overset{def}{=} \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}},$$

where $U_{ij} \overset{def}{=} U_i \cap U_j$. By part 3 of the proof of Theorem $II.3.3$ in [Har77] these morphisms glue. $\square$

By Theorem 2.3.12, we see line bundles on $\mathcal{O}_{\mathbb{P}^N}$ are unique up to degrees. So, it follows that as groups

$$\operatorname{Pic}(\mathbb{P}^N) \cong \mathbb{Z}.$$

2.3.2 Examples of Bundles

To develop a theory of a canonical bundle on a curve, and to compute it, we will need four standard kinds of vector bundles which exist on many kinds of schemes. These are the tangent and cotangent sheaves, the normal bundle, and finally the canonical bundle itself.

The Sheaf of Differentials

As usual in this section, we begin with some facts about graded rings.

Definition 2.3.13 ([Har77, page 172]). *Let A be a commutative ring with 1, let B be an A-algebra and let M be a B-module. An A-**derivation** of B into M is a map $d : B \to M$ such that*

1. $d(b + b') = d(b) + d(b')$ *for all* $b, b' \in B$,
2. $d(bb') = bd(b') + b'd(b)$ *for all* $b, b' \in B$, *and*
3. $d(a) = 0$ *for all* $a \in A$.

Now we may formally define a module of differentials in the "right" way to extend the definition to schemes.

Definition 2.3.14 ([Har77, page 172]). *Let A be a commutative ring with 1 and let B be an A-algebra. Define the **module of relative differential forms** of B over A to be the B-module $\Omega_{B/A}$ equipped with the A-derivation $d : B \to \Omega_{B/A}$ which satisfies* *the universal property that for any B-module M and any A-derivation $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.*

Example 2.3.15 ([Har77, Example II.8.2.1])**.** *Let X be a scheme of dimension n over $\mathbb{C}$, and let $B \stackrel{def}{=} \mathbb{C}[x_0, \cdots, x_{n-1}]$. Then $\Omega_{B/\mathbb{C}}$ is the free B-module of rank n generated by $dx_0, \cdots, dx_{n-1}$, and we denote by Ω_X the **sheaf of differential 1-forms** on X, with associated module $\Omega_{B/\mathbb{C}}$.*

Any actual treatment of duals of sheaves is besides the point in this discussion, so we state a definition of a tangent sheaf so that we can connect differentials and the normal bundle, which we turn to next.

Definition 2.3.16. *Let X be a scheme over an algebraically closed field $\mathbb{F}$. The **tangent bundle** T_X to X is the bundle $T_X \stackrel{def}{=} \Omega_X^\vee$.*

Normal Bundle

At first glance, the normal bundle appears to be simply yet another sheaf of modules on a scheme with a particularly unfriendly looking quotient definition. However, we have carefully picked an exceptionally friendly kind of scheme: a complete intersection, to compute the normal bundle for.

Formally, we define the vector bundle of normal vectors to a subscheme as follows.

Definition 2.3.17 ([EH16, page 50])**.** *Suppose $X \subset Y$ is an inclusion of schemes over a field $\mathbb{F}$. Then there is an inclusion of bundles $T_X \subset T_Y|X$ and the quotient bundle*

$$N_X \stackrel{def}{=} T_Y|_X/T_X$$

*is the **normal bundle** to X in Y.*

The connection between determinants of bundles, complete intersections of hypersurfaces, and curves all hinges on the following theorem.

Theorem 2.3.18. *Suppose $X \subset \mathbb{P}^N$ is a curve which is the complete intersection of hypersurfaces $D_1, \cdots, D_r \subset \mathbb{P}^N$. Then*

$$N_X = \mathcal{O}_{\mathbb{P}^N}(D_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^N}(D_r).$$

The Canonical Bundle

I decline to comment on why the canonical bundle is so named.

Our definition of a canonical bundle has two forms: the explicit catch-phrase "top exterior power of the sheaf of differentials" definition for computations, and for experts, the derived functor definition.

Definition 2.3.19 ([Har77, page 180])**.** *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension n. We define the **canonical bundle** $\omega = \omega_X$, a line bundle on X, by*

$$\omega \overset{def}{=} \bigwedge^n \Omega_X^1$$

where $N = \dim H^0(X, \omega_X) - 1$ and Ω_X^1 is the sheaf of regular differential one-forms on X from Definition 2.3.14.

Definition 2.3.20 ([Rei19, Lecture 10])**.** *Let $X \subset \mathbb{P}^N$ be some projective connected variety of dimension n. The **canonical bundle** ω_X is*

$$\omega_X = \mathrm{Ext}^{N-n}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N}),$$

where Ext *is the derived functor of sheaf* $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^N}}(\cdot, \omega_{\mathbb{P}^N})$.

If the variety X from Definition 2.3.20 is also non-singular, then $\omega_X \cong \Omega^n_X$. This line bundle is so special that the associated divisor has a distinguished name.

Definition 2.3.21. *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension n. The **canonical divisor** K_X on X is the (Cartier) divisor associated to ω_X.*

We make one final restriction on the kind of curves which we consider from this point forward.

Definition 2.3.22. *Let $X \subset \mathbb{P}^N$ be a curve and suppose that ω_X is very ample. Then we call X a **canonical curve**.*

A feature of projective curves X is that we can compute the coordinate ring of X by means of the coordinate ring of $\mathbb{P}^N_{\mathbb{F}} = \mathrm{Proj}\,\mathbb{F}[x_0, \cdots, x_N]$. In particular, we can map to this ring by means of the map associated to ω_X given in Definition 2.3.6.

A central object in the study of curves is the following ring associated to the canonical bundle.

Definition 2.3.23 ([VZB22, page 1])**.** *Let X be a scheme over an algebraically closed field $\mathbb{F}$. The **canonical ring** of X is the ring*

$$R = R(X, \omega_X) = \bigoplus_{n \geqslant 0} H^0(X, \omega_X^{\otimes n}).$$

Remark 2.3.24. *It is a Fields-medal winning result that the canonical ring is finitely generated, and the proof in full generality is too involved for these notes, which are concerned with the more classical theorem mentioned below.*

Remark 2.3.25. *For $\mathcal{L}$ any very ample line bundle on a scheme X over an algebraically closed field k we can define a section ring of $\mathcal{L}$ analogously to the canonical ring defined above. One particularly relevant example for these notes is the Arbarello-Sernesi module of X and $\mathcal{L}$ a line bundle on X which is the graded module*

$$\bigoplus_{q\in\mathbb{Z}} H^0(X, \omega_X \otimes \mathcal{L}^{\otimes q}),$$

which can be used, as in [GL85] with $\mathcal{L} = \omega_X$, to prove the theorem of Enriques, Babbage and Petri, known as Petri's theorem. Some other examples of explicit computations of section rings are [O'D15] for divisors with $\mathbb{Q}$-coefficients on $\mathbb{P}^1$, [CFO24] which generalizes [O'D15] to elliptic curves, and [LRZ16] which computes (log) spin canonical rings of curves in all genera. A comprehensive summary of canonical and log canonical rings of curves in all genera is found in [VZB22, Chapter 2].

Now that we have a basic sense of what a canonical bundle is we turn the discussion to computing it in the case of curves which are complete intersections of hypersurfaces.

Lemma 2.3.26 ([Sha13, Shafarevich's Lemma]). *Let X be a purely n-dimensional, non-singular, smooth, projective, algebraic variety over $\mathbb{C}$. Locally, the canonical bundle on X has form $\omega = f(dx_1 \wedge \cdots \wedge dx_n)$, where $x_1, \cdots, x_n$ are some local parameters and f is some regular function.*

We first compute the canonical bundle on the scheme $\mathbb{P}^N = \operatorname{Proj} \mathbb{C}[x_0, \cdots, x_N]$.

Theorem 2.3.27 ([Vak02a]). $\omega_{\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^N}(-N-1)$.

Proof. For readability this proof is restricted to the case when $N = 2$.

Let $\mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2]$ and consider some charts

$$\begin{array}{llcll} U_0 & = \{x_0 \neq 0\} & \text{coordinates} & (u_1, u_2), & \begin{cases} u_1 \overset{def}{=} \frac{x_1}{x_0} \\ u_2 = \frac{x_2}{x_0} \end{cases} \\ U_1 & = \{x_1 \neq 0\} & -- & (v_0, v_2), & v_i \overset{def}{=} \frac{x_i}{x_1} \\ U_2 & = \{x_2 \neq 0\} & -- & (w_0, w_1), & w_i \overset{def}{=} \frac{x_i}{x_2}. \end{array}$$

By Shafarevich's Lemma 2.3.26, sections of $\omega_{\mathbb{P}^2}$ over U_0 have form $f(u_1, u_2) du_1 \wedge du_2$ for some $f \in \mathcal{O}_{\mathbb{P}^2}$, so consider the section $du_1 \wedge du_2$ in particular. Away from U_0 there is one location in $\mathbb{P}^2$ where we want to make sense of our section $du_1 \wedge du_2$, namely the divisor $x_0 = 0$. In the coordinates of the chart U_1, which contains the divisor $x_0 = 0$, we observe with some elementary calculus that

$$du_1 \wedge du_2 = \left(\frac{-1}{u_0^2} du_0 \right) \wedge \left(\frac{u_0 du_2 - u_2 du_0}{u_0^2} \right),$$

and since $e_i \wedge e_i \overset{def}{=} 0$ for any vector e_i, we conclude

$$du_1 \wedge du_2 = \frac{-1}{u_0^3} du_0 \wedge du_2.$$

Since $\frac{-1}{u_0^3}$ has a pole of order 3 on $u_0 = 0$ as desired we are done. □

We will use a version of the adjunction formula to compute the canonical bundle of our complete intersections.

Theorem 2.3.28 (Adjunction Formula)**.** *If $X \subset \mathbb{P}^N$ is a smooth subscheme with normal bundle N_X then*

$$\omega_X \cong \omega_{\mathbb{P}^N}|_X \otimes \det(N_X).$$

2.4 A Genus Formula for Complete Intersections of Surfaces

Now we have the tools to state and prove a genus formula for complete intersections of hypersurfaces in a projective space. We restrict ourselves to hypersurfaces in $\mathbb{P}^4$ for this section to make the notation concrete and as accessible as possble.

Theorem 2.4.1. *Let $X \subseteq \mathbb{P}^4$ be the complete intersection of smooth degree d_1, d_2, d_3 hypersurfaces D_1, D_2 and $D_3 \subset \mathbb{P}^4$. Then X is a curve of genus*

$$g = \frac{(d_1 + d_2 + d_3 - 5)d_1 d_2 d_3 - 2}{2}.$$

Proof. Recall that by Exercise $I.2.17.b$ in [Har77], X is a set-theoretic complete intersection and therefore a curve since it is the intersection of 3 hypersurfaces in $\mathbb{P}^4$, i.e. a variety of dimension 1 per Definition 2.2.3. By the Adjunction formula 2.3.28, we compute

$$\omega_X = \omega_{\mathbb{P}^4}|_X \otimes \det(N_X).$$

Using Theorem 2.3.18 and Theorem 2.3.27 we see

$$\omega_X = \mathcal{O}_{\mathbb{P}^4}(-5)|_X \otimes \det\left[\mathcal{O}_{\mathbb{P}^4}(D_1)|_X \oplus \mathcal{O}_{\mathbb{P}^4}(D_2)|_X \oplus \mathcal{O}_{\mathbb{P}^4}(D_3)|_X\right].$$

We can compute the determinant with Theorem 2.1.6, and since the $\mathcal{O}_{\mathbb{P}^4}(D_i)|_X$ are line bundles for $1 \leqslant i \leqslant 3$, Definition 2.1.5 becomes $\det(\mathcal{O}_{\mathbb{P}^4}(D_i)|_X) = \bigwedge^1 \mathcal{O}_{\mathbb{P}^4}(D_i)|_X =$

$\mathcal{O}_{\mathbb{P}^4}(D_i)|_X$ for each i, so we get

$$\omega_X = \mathcal{O}_{\mathbb{P}^4}(-5)|_X \otimes \mathcal{O}_{\mathbb{P}^4}(D_1)|_X \otimes \mathcal{O}_{\mathbb{P}^4}(D_2)|_X \otimes \mathcal{O}_{\mathbb{P}^4}(D_3)|_X$$

by Theorem 2.1.6. Since $\mathrm{Pic}(\mathbb{P}^N) \cong \mathbb{Z}$ so line bundles are unique up to degrees, using Serre twist notation 2.3.12 and the fact that restrictions commute with tensors since restriction is a right adjoint functor, we rewrite

$$\omega_X = \mathcal{O}_{\mathbb{P}^4}(-5) \otimes \mathcal{O}_{\mathbb{P}^4}(d_1) \otimes \mathcal{O}_{\mathbb{P}^4}(d_2) \otimes \mathcal{O}_{\mathbb{P}^4}(d_3)|_X.$$

Finally, making use of the convenient notation choice above and the Picard group again,

$$\omega_X = \mathcal{O}_{\mathbb{P}^4}(d_1 + d_2 + d_3 - 5)|_X = \mathcal{O}_X(d_1 + d_2 + d_3 - 5).$$

By Theorem 2.3.12 we have an isomorphism

$$\mathcal{O}_{\mathbb{P}^4}(d_1 + d_2 + d_3 - 5)|_X \cong \mathcal{O}_{\mathbb{P}^4}((d_1 + d_2 + d_3 - 5)H)|_X$$

for H any hyperplane divisor. Being a hyperplane divisor, H will intersect X, which has degree $d_1d_2d_3$ by Corollary 2.2.5, exactly $\deg(X) = d_1d_2d_3$ times, so that

$$\deg(\mathcal{O}_X(d_1 + d_2 + d_3 - 5)) = (d_1 + d_2 + d_3 - 5)d_1d_2d_3.$$

By Riemann-Roch and Corollary 2.2.5 we compute

$$\begin{aligned} \deg(K_X) &= \deg(\mathcal{O}_{\mathbb{P}^4}(d_1+d_2+d_3-5)|_X) &= 2g-2 \\ & (d_1+d_2+d_3-5)d_1d_2d_3 &= 2g-2, \end{aligned}$$

so

$$g = \frac{(d_1+d_2+d_3-5)d_1d_2d_3-2}{2}. \tag{2.4.1}$$

□

Corollary 2.4.2. *The complete intersection of* 3 *distinct smooth quadrics in* $\mathbb{P}^4$ *is a curve of genus* 5.

Proof. For each i, we have $d_i = 2$ and so by our formula 2.4.1 we compute $g = 5$. □

2.5 Explicit Syzygies of Homogeneous Ideals

We also want to show that if X is a genus $g \geqslant 4$ curve (over $\mathbb{C}$ in order to simplify, any algebraically closed field of characteristic 0 works as well), then the canonical ring of X has form $R \cong \mathbb{C}[x_1, \cdots, x_g]/I$, where I is generated by (exactly 3 when $g = 5$) quadrics. This is the other direction of Petri's theorem's "if and only if"-type statement.

As an R-ideal, I naturally has the structure of an R-module, and in fact is finitely generated. However, more information is needed than simply the generators, say some $f_1, \cdots, f_n$, for I. In particular there are nontrivial relations among those generators, which form a set called the **(first) syzygies**, denoted $\mathrm{Syz}(f_1, \cdots, f_n)$ following the notation from [CLO05, chapter 6]. It turns out that $\mathrm{Syz}(f_1, \cdots, f_n)$ is itself an R-module, say with generators $g_1, \cdots, g_m$, and there is an R-module of relations among the g_i, denoted $\mathrm{Syz}(g_1, \cdots, g_m)$, which is the module of **(second) syzygies** for I. Proceeding in this way one defines a sequence of successive syzygy modules for I which is called a **resolution**. Our goal will be to explicitly write down the first syzygies corresponding to the quadrics whose complete intersection is (the image under the canonical embedding in $\mathbb{P}^{g-1}$ of) X.

Let $\varphi : X \to \mathbb{P}^{g-1}$ be the map obtained from global sections of the canonical bundle

$$p \mapsto [s_1(p), \cdots, s_g(p)]$$

and let $x_1, \cdots, x_g \in X$ be some closed points in general position. Then consider a basis $\varphi_1, \cdots, \varphi_g$ of $H^0(X, \omega_X)$ such that $\varphi_i(x_j) \neq 0$ if and only if $i = j$. By the uniform position theorem in [ACG11, Section 3] and the geometric Riemann-Roch

$$\dim H^0(X, K(-x_1 - \cdots - \hat{x_i} - \cdots - x_g)) = 1,$$

where $\hat{x_i}$ means that point is excluded, φ_i is taken to be the generator for each i and $K = K_X$ is a canonical divisor on X. As a section of K

$$\begin{cases} \varphi_i(x_i) \neq 0, \\ \varphi_i(x_j) = 0, \quad i \neq j \end{cases}$$

so the φ_i form a basis for $H^0(X, K)$. The assumption that the points x_i are in general position also means the divisors (φ_i) are supported at $2g-2$ distinct points with pairwise disjoint support. Note that for any relation

$$\sum \lambda_i \varphi_i = 0,$$

evaluating at x_i gives $\lambda_i = 0$. It is also worth noting that this choice of basis is not arbitrarily restrictive in the sense of the following Lemma.

Lemma 2.5.1. *Let X be a genus $g \geqslant 4$ canonical, non-hyperelliptic, smooth, irreducible, complex algebraic curve. Let $\varphi : X \to \mathbb{P}^{g-1}$ be the map obtained from global sections of the canonical bundle*

$$p \mapsto [s_1(p), \cdots, s_g(p)]$$

and let $x_1, \cdots, x_g \in X$ be some closed points in general position. Suppose $\varphi_1, \cdots, \varphi_g$ form a basis for $H^0(X, \omega_X)$ such that $\varphi_i(x_j) \neq 0$ if and only if $i = j$. Then given any basis $\eta_1, \cdots, \eta_g$ for $H^0(X, \omega_X)$, there exist some $a_{i,j} \in \mathbb{C}$ such that $\varphi_i = \sum_{k=1}^{g} a_{i,k}\eta_k$.

Proof. Let $\eta_1, \cdots, \eta_g$ be a basis for $H^0(X, \omega_X)$. Since the data of $H^0(X, \omega_X)$ is some cover by affine opens $(U_i \to X)_{i\in\Lambda}$ with sections $s_i \in \omega_X(U_i)$ compatible over intersections, for any $x \in X$, the η's globally generate $H^0(X, \omega_X)$ in the sense that

$$\omega_{X,x} = \operatorname{span}\{(\eta_1)_x, \cdots, (\eta_g)_x\}.$$

One of the rational sections $(\eta_i)_x$ is a generator for the localization $\omega_{X,x}$ at x. Suppose for each of $x_1, \cdots, x_g \in X$ some closed points in general position, that $\alpha_1, \cdots, \alpha_g$

generate $\omega_{X,x_1}, \cdots, \omega_{X,x_g}$ respectively. Then

$$\begin{array}{lllll}
(\eta_1)_{x_1} = r_1\alpha_1 & (\eta_1)_{x_2} = r_2\alpha_2 & \cdots & (\eta_1)_{x_g} = r_g\alpha_g \\
(\eta_2)_{x_1} = s_1\alpha_1 & (\eta_2)_{x_2} = s_2\alpha_2 & \cdots & (\eta_2)_{x_g} = s_g\alpha_g \\
\qquad\vdots & & & \qquad\vdots \\
(\eta_g)_{x_1} = t_1\alpha_1 & & \cdots & (\eta_g)_{x_g} = t_g\alpha_g
\end{array}$$

for some $r_1, s_1, \cdots, t_1 \in \mathcal{O}_{X,x_1},\ r_2, s_2, \cdots, t_2 \in \mathcal{O}_{X,x_2},\ r_g, s_g, \cdots, t_g \in \mathcal{O}_{X,x_g}$ and so on. Recall that each of the local rings $\mathcal{O}_{X,x_i}$ is a discrete valuation ring with a unique maximal ideal the uniformizer at x_i. Since ω_{X,x_i} is generated by α_i for each i,

$$\langle r_i, s_i, \cdots, t_i \rangle = \mathcal{O}_{X,x_i}$$

so one of $r_i, s_i, \cdots, t_i \in \mathcal{O}_{X,x_i}^{\times}$. Suppose for some $a_{i,1}, \cdots, a_{i,g} \in \mathbb{C}$ not all 0 that

$$(a_{i,1}\eta_1 + a_{i,2}\eta_2 + \cdots + a_{i,g}\eta_g)(x_j) = 0$$

for some $j \neq i$. At the stalk

$$(a_{i,1}\eta_1 + a_{i,2}\eta_2 + \cdots + a_{i,g}\eta_g)_{x_j} = \left[a_{i,1}(r_j(x_j)) + a_{i,2}(s_j(x_j)) + \cdots + a_{i,g}(t_j(x_j))\right]\alpha_j$$

so without loss of generality if r_j is the unit, since $a_{i,1}r_j + \cdots + a_{1,g}t_j = 0$,

$$a_{i,1} = -r_j^{-1}(x_j)\left[a_{i,2}s_j(x_j) + \cdots + a_{i,g}t_j(x_j)\right].$$

In particular the solution lies in $\mathbb{C}$. Indeed $r_j, s_j, \cdots, t_j \in \mathcal{O}_{X,x_j}$ so the evaluations

$r_j(x_j), \cdots, s_j(x_j) \in \mathcal{O}_{X,x_j}/\mathcal{M} = \kappa(X)$, where $\mathcal{M}$ is the uniformizer at x_2 and $\kappa(X) = \kappa(x_j)$ is the residue field of the curve at the stalk. So since $s_j, \cdots, t_j$ vanish to nonnegative order at x_j as localizations of a global section to an affine open, and r_j by assumption of being a unit is nonvanishing at x_j,

$$r_j \in \mathcal{O}_{X,x_j}^{\times} \Rightarrow r_j(x_j) \in (\mathcal{O}_{X,x_j}/\mathcal{M})^{\times} = \mathbb{C}^{\times}$$

and each of $s_j(x_j), \cdots, t_j(x_j)$ lie in a finite extension of $\mathbb{C}$. Therefore, each is a complex number since $\mathbb{C}$ is algebraically closed, so there are no such nontrivial extensions of $\mathbb{C}$. □

Next we consider the relations in our chosen basis $\{\varphi_i\}$ for $H^0(X, \omega_X)$. Ultimately we will give bases for each graded component of the canonical ideal I of X in $\mathbb{P}^{g-1}$ as in [Mum99, page 237]. Consider the maps

$$\psi_n : H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(n)) \to H^0(X, \omega_X^{\otimes n})$$

given by restriction and let $X_1, \cdots, X_g$ be a basis for $H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ defined by

$$X_i = \psi_1^{-1}(\varphi_i),$$

so that the X_i act like homogeneous coordinates.

Example 2.5.2 ([ACG11, page 125])**.** *Given $P = P(X_1, \cdots, X_g) \in H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(n))$ say that $\overline{P} = \psi_n(P)$. Changing coordinates in this manner when $n = 3$ we have for example:*

$$\overline{X_1^2 X_3} = \varphi_1^2 \varphi_3.$$

Let $D = x_3 + \cdots + x_g \in \mathrm{Div}(X)$. The general position of the x_i means

$$\dim H^0(X, \omega_X(-D)) = 2,$$

where the vector space has a basis φ_1 and φ_2. Since the support of the (φ_i) are pairwise disjoint, the pencil $|\omega_X(-D)|$ is **base-point free**. Each vector space in the filtration

$$H^0(X, \omega_X^n) \supset H^0(X, \omega_X^n(-D)) \supset \cdots \supset H^0(X, \omega_X^n((-n+1)D))$$

has codimension $g-2$ in the previous, where $n-1 \geqslant s \geqslant 1$ since by Riemann-Roch, for each s we have

$$h^0(X, \omega_X^n(-sD)) = (2n-1)(g-1) - s(g-2).$$

To actually write Petri's equations for each s there must be n-canonical forms in $H^0(X, \omega_X^n(-sD))$ which are linearly independent modulo $H^0(X, \omega_X^n((-s-1)D))$ as this allows us to form a basis for the canonical ring.

Lemma 2.5.3 ([ACG11, Base-point free pencil trick])**.** *Let C be a smooth curve, let L be an invertible sheaf on C and let $\mathcal{F}$ be a free $\mathcal{O}_C$-module. Suppose s_1 and s_2 are linearly independent sections of L and denote the subspace of $H^0(C, L)$ which they generate V. Then the map*

$$\phi_{2,2} : V \otimes H^0(C, \mathcal{F}) \to H^0(C, \mathcal{F} \otimes L)$$

given by

$$s_1 \otimes t_2 - s_2 \otimes t_1 \mapsto s_1 t_2 - s_2 t_1$$

has kernel

$$\ker \phi_{2,2} \cong H^0(C, \mathcal{F} \otimes L^{-1}(B)),$$

where B is the base locus of the pencil spanned by s_1 and s_2.

The application the Base-point free pencil trick 2.5.3 relevant to the Petri equations is our computation of

$$\ker \phi_{n,s} \cong H^0(C, \omega_C^{n-2}((-s+2)D))$$

in the case when C is our smooth genus $g \geqslant 4$ curve and $\phi_{n,s}$ for $n-1 \geqslant s \geqslant 1$ is the cup-product map

$$\phi_{n,s} : H^0(C, \omega_C^{n-1}((-s+1)D)) \otimes H^0(C, \omega_C(-D)) \to H^0(C, \omega_C^n(-sD))$$

from [ACGH85, 3.3].

We start with an inductive desciption of bases for the vector spaces $H^0(X, \omega_X^n)$ for each n as follows. The map

$$\phi_{2,1} : H^0(X, \omega_X) \otimes H^0(X, \omega_X(-D)) \to H^0(X, \omega_X^2(-D))$$

is surjective by Lemma 2.5.3 so

$$\varphi_1^2,\ \varphi_1\varphi_2,\ \varphi_2^2,\ \varphi_1\varphi_i,\ \varphi_2\varphi_i,$$

where $3 \leqslant i \leqslant g$, form a basis for $H^0(X, \omega_X^2(-D))$. At the top of the tower

$$H^0(X, \omega_X^2) \supset H^0(X, \omega_X^2(-D)),$$

the $\varphi_3^2, \cdots, \varphi_g^2$ are differentials in $H^0(X, \omega_X^2)$ which are linearly independent modulo $H^0(X, \omega_X^2(-D))$ and since $\operatorname{codim}(H^0(X, \omega_X^2(-D))$ in $H^0(X, \omega_X^2)) = g - 2$ the basis for $H^0(X, \omega_X^2)$ is

$$\begin{array}{cccl}
\varphi_1^2, \varphi_1\varphi_2, \varphi_2^2 & | & | & \\
\varphi_1\varphi_i & | & | & \text{basis of } H^0(X, \omega_X^2(-D)) \\
\varphi_2\varphi_i & | & | & \\
\varphi_3^2, \cdots, \varphi_g^2 & | & & \quad \text{basis of } H^0(X, \omega_X^2).
\end{array}$$

Writing down all of the differentials in each homogeneous order n, some nontrivial relations begin to arise between them. For example, for all $3 \leqslant i, k \leqslant g$ where $i \neq k$, $\varphi_i\varphi_k \in H^0(X, \omega_X^2(-D))$ and in particular vanishes at x_1 and x_2. In [Mum99, page 240] Mumford concisely describes these relations

$$\varphi_i\varphi_j = \sum_{k=3}^{g} \alpha_{ijk}(\varphi_1, \varphi_2)\varphi_k + \nu_{ij}\varphi_1\varphi_2,$$

and in $H^0(X, \omega_X^3)$ in particular

$$\eta_i - \eta_j = \sum_{k=3}^{g} \alpha'_{ijk}(\varphi_1, \varphi_2)\varphi_k + \nu'_{ij}\varphi_1^2\varphi_2 + \nu''_{ij}\varphi_1\varphi_2^2,$$

where the α are linear, α' are quadratic and ν's are scalars repectively. In particular the homogeneous degree 2 equations

$$f_{ij} = x_i x_j - \sum_{k=3}^{g} \alpha_{ijk}(x_1, x_2)x_k - \nu_{ij}x_1x_2,$$

and the degree 3 equations

$$g_{ij} = (\mu_i x_1 - \lambda_i x_2)x_i^2 - (\mu_j x_1 - \lambda_j x_2)x_j^2 - \sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{ij}x_1^2x_2 - \nu''_{ij}x_1x_2^2,$$

where the $3 \leqslant i, j \leqslant g$, and $i \neq j$ are generators of the ideal of X in $\mathbb{P}^{g-1}$. In other words the f_{ij} all vanish on X in $\mathbb{P}^{g-1}$ and are exactly the subvariety-defining equations guaranteed by Petri's theorem. To be rigorous, these

$$\frac{(g-2)(g-3)}{2}$$

linearly independent elements of I_2 match the dimension of I_2 which we expect from Max Noether's theorem, so indeed the f_{ij} form a basis.

Example 2.5.4. *The full list of these equations when $g = 5$ is*

$$f_{34}, \quad f_{35}, \quad f_{43}, \quad f_{45}, \quad f_{53}, \quad f_{54}$$
$$g_{34}, \quad g_{35}, \quad g_{43}, \quad g_{45}, \quad g_{53}, \quad g_{54}$$

but there are some relations among them.

We describe relations among our generators in the next Lemma.

Lemma 2.5.5 ([Mum99, page 240])**.** *Let X be a genus $g \geqslant 4$ canonical, non-hyperelliptic, smooth, irreducible, complex algebraic curve. There are syzygies*

1. $f_{ij} = f_{ji}$

2. $g_{ij} + g_{jk} = g_{ik}$.

3. $x_k f_{ij} - x_j f_{ik} + \sum_{\substack{l=3 \\ l \neq k}}^{g} \alpha_{ijl} f_{kl} - \sum_{\substack{l=3 \\ l \neq k}}^{g} \alpha_{ikl} f_{jl} = \rho_{ijk} g_{jk}$,
   where $3 \leqslant i, j, k \leqslant g$, i, j, k *are distinct, and the* ρ_{ijk} *are scalars symmetric in* i, j *and* k,

which generate the components of the homogeneous ideal of X in its canonical embedding $I_{X/\mathbb{P}^{g-1},2}$ and $I_{X/\mathbb{P}^{g-1},3}$ respectively.

Proof. This is a proof of only the second syzygy. The first is trivial and the third requires more discussion.

$$\begin{aligned} g_{ij} + g_{jk} &= (\mu_i x_1 - \lambda_i x_2)x_i^2 - (\mu_j x_1 - \lambda_j x_2)x_j^2 - \textstyle\sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{ij} x_1^2 x_2 - \nu''_{ij} x_1 x_2^2 \\ &\quad + (\mu_j x_1 - \lambda_j x_2)x_j^2 - (\mu_k x_1 - \lambda_k x_2)x_k^2 - \textstyle\sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{jk} x_1^2 x_2 - \nu''_{jk} x_1 x_2^2 \\ &= (\mu_i x_1 - \lambda_i x_2)x_i^2 - (\mu_k x_1 - \lambda_k x_2)x_k^2 - \textstyle\sum_{k=3}^{g} \alpha'_{ijk}(x_1, x_2)x_k - \nu'_{ik} x_1^2 x_2 - \nu''_{ik} x_1 x_2^2 \\ &= g_{ik}, \end{aligned}$$

where $\nu'_{ik} = \nu'_{ij} + \nu'_{jk}$ and $\nu''_{ik} = \nu''_{ij} + \nu''_{jk}$. $\square$

In order to spare writing longer lists and make the notation more readable, we suppose $g(X) = 5$ as this suffices to illustrate the point. The first two kinds of syzygy in Lemma 2.5.5 reduces the number of relations per the following table when $g = 5$:

type (1)	type (2)
$f_{34} = f_{43}$	$g_{34} + g_{45} = g_{35}$
$f_{35} = f_{53}$	$g_{35} + g_{54} = g_{34}$
$f_{45} = f_{54}$	$g_{45} + g_{53} = g_{43}$
	$g_{43} + g_{35} = g_{45}$
	$g_{53} + g_{34} = g_{54}$
	$g_{54} + g_{43} = g_{53}$

which leaves only the following generators for the ideal

$$f_{34}, f_{35}, f_{45}, g_{34}, g_{35}, g_{45}$$

subject to the relations

$$\rho_{354} g_{34} = x_4 f_{35} - x_5 f_{34} + \sum_{\substack{l=3 \\ l \neq 4}} \alpha_{35l} f_{4l} - \sum_{\substack{l=3 \\ l \neq 4}} \alpha_{34l} f_{5l},$$

$$\rho_{345} g_{35} = x_5 f_{34} - x_4 f_{35} + \sum_{\substack{l=3 \\ l \neq 5}} \alpha_{34l} f_{5l} - \sum_{\substack{l=3 \\ l \neq 5}} \alpha_{35l} f_{4l},$$

and

$$\rho_{435} g_{45} = x_5 f_{43} - x_3 f_{45} + \sum_{\substack{l=3 \\ l \neq 5}} \alpha_{43l} f_{5l} - \sum_{\substack{l=3 \\ l \neq 5}} \alpha_{45l} f_{3l}.$$

Now that we have illustrated explicit relations, we return to the consideration of general genus $g \geqslant 4$ curves. There are several cases with different minimal sets of relations. Either $\rho_{ijk} = \alpha_{ijk} = 0$ whenever i, j, k are distinct, in which case our curve

X is either trigonal or in the genus 6 case may be a nonsingular plane quintic; or $\{3, \cdots, g\} = I_1 \cup I_2$, where for all $j \in I_1$ and $k \in I_2$ there exists an i with $\rho_{ijk} \neq 0$ and $\alpha_{ijk} \neq 0$ and such that the ideal of X is generated by the f_{ij} alone. Finally, we can state and prove our version of Petri's result about the canonical ideal of a genus 5 curve.

Theorem 2.5.6. *Let X be a genus 5 canonical, non-hyperelliptic, smooth, irreducible, complex algebraic curve. The syzygies f_{34}, f_{35} and f_{45} generate the canonical ideal of X its canonical embedding in $\mathbb{P}^4$.*

Proof. Consider any partition of $\{3, 4, 5\}$ which includes at least one nonempty subset and the set-theoretic complement of that the first component. If at least some $\rho_{ijk} \neq 0$ then g_{ik} is determined by the f_{ij}. If every g were to be 0 the result also follows. $\square$

Chapter 3

Stacks and How we Compute their Canonical Rings

In this chapter we consider the problem of doing a calculation of the canonical ring in the spirit of Petri's theorem but for a stack rather than a scheme. As we will see, in the case of tamely ramified Deligne-Mumford stacky curves that we are interested in, our stacks behave just like schemes with some finite number of fractional points. This makes the combinatorics of the canonical ring slightly more complicated than in Petri's theorem, but is a quite well understood question with lots of existing theory. We first describe carefully what a stack is and then provide some references to existing techniques and results for computing canonical rings stacky curves.

3.1 Why do we use Stacks?

We preview our working theory of stacks with some motivation for this addition level of technicality. In doing so, we will mention several facts that we return to in later

Sections and Chapters, and some that are not explicitly covered elsewhere in this document. Without further ado, the reason that we use stacks is because of how uniquely suited they are for describing the geometry of modular forms.

Modular forms as in [DS05, Definition 1.2.3] and Definition 6.1.5 can always be made (in the non-Drinfeld and Drinfeld case respectively) as global sections of line bundles. However, the dimension formulas (see e.g. [DS05, Chapter 3]) do not agree with the dimensions of the global sections of these line bundles, i.e. we do not get $M_k(\Gamma) = H^0(X(\Gamma), L^{\otimes k})$. Modular forms (including Drinfeld ones) can always be treated as sections of line bundles without referring to the stacky structure of the modular curves where the modular form lives. However, it is not true that $H^0(X, L^k) = M_k$ where

(X = moduli space, L = appropriate line bundle, M = vector space of modular forms)

without either modifying L, taking a subspace of H^0 or replacing X with a stack whose coarse space is X.

This problem is more than just the dimension counts, it is the full graded ring.

Modular forms *are* sections of line bundles, the point is that just computing the graded ring of sections of powers of a single line bundle does not give the correct ring of modular forms, at least without modification. On the other hand, treating this canonical ring as a stacky canonical ring - meaning computing global sections of powers of a line bundle *on a stack* - does recover the correct graded ring structure.

The difference between $H^0(X, L)$ and $H^0(\mathscr{X}, \mathscr{L})$ for a scheme and a stack respectively is subtle, but we can phrase things in terms of divisors. If $\mathscr{L} = \mathcal{O}(D)$ for a divisor on $\mathscr{X}$ (see Section 3.3), then $H^0(\mathscr{X}, D) = H^0(X, \lfloor D \rfloor)$, where $\lfloor D \rfloor$ denotes the floor of D. Explicitly, D is a sum of irreducible divisors, some of which may be stacky. For such an irreducible divisor Z, $\lfloor Z \rfloor = \lfloor 1/\#G \rfloor Z$, where G is the automorphism group along Z.

There are floors in the formulas for (dimensions of algebras of) modular forms, so it looks like something jumpy and discontinuous is happening. We know modular forms are functions on the j-line (see [DS05, page 7] or [Gek86, Example V.3.6]), so what is going on? Certain isomorphism classes (of say elliptic curves or Drinfeld modules of rank 2 respectively) are "fatter" than the rest. For example, elliptic curves with $j = 0$ (respectively $j = 1728$) have 6 automorphisms (respectively 4) instead of just the usual hyperelliptic involution. Here is our "jumpiness." Counting properly, i.e. treating those j values as 1/3 and 1/2 we get a continuous looking formula.

3.2 What is a Stack

Thanks to Yoneda's lemma we may introduce stacks in terms of a familiar language to a geometer. A stack is a category fibered over some other category all of whose morphisms are isomorphisms (a groupoid), and which satisfies a descent condition, an analog of the ordinary sheaf condition. A useful practical reference for doing work with stacks is [Alp23], where Alper uses this perspective to introduce stacks via a “pre-stack” much like a presheaf. We first state this analogy between sheaves and stacks as a means to organize this chapter, and then we will make it precise.

1-category	2-category
functor/pre-sheaf	fibered category
separated pre-sheaf	pre-stack
sheaf	stack
algebraic space / scheme	algebraic stack
variety	algebraic stack of finite type over a field

We take the functor of points perspective when working with stacks as this allows us to make very explicit calculations and deal with the coming category theory in a way which resembles how we learn about sheaves.

3.2.1 Functor of Points, Yoneda, Sites, Groupoids

Let $\mathscr{C}$ be a category, and let $\hat{\mathscr{C}} \overset{def}{=} \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathrm{Set})$ denote the category of functors whose objects are functors and whose morphisms are natural transformations. Then we say the functor $h : \mathscr{C} \to \hat{\mathscr{C}}$ sending an object $X \in \mathscr{C}$ to the functor $h_X \in \hat{\mathscr{C}}$ defined by $h_X(Y) \overset{def}{=} \mathrm{Hom}_{\mathscr{C}}(Y, X)$ and mapping morphisms as in the following diagram:

$$\begin{tikzcd} Y' \arrow[d, "g"] \arrow[dr] & \\ Y \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(Y, X) \ni f \arrow[d] & f \arrow[d, mapsto] \\ \mathrm{Hom}(Y', X) \ni f \circ g, & \end{tikzcd}$$

is the **functor of points** on $\mathscr{C}$. We say a functor $F \in \hat{\mathscr{C}}$ is **representable** if there is some $X \in \mathscr{C}$ and an isomorphism $h_X \xrightarrow{\sim} F$, i.e. for each $Y \in \mathscr{C}$ there is an isomorphism $h_X(Y) \xrightarrow{\sim} F(Y)$ compatible with compositions. A map $F' \to F \in \hat{\mathscr{C}}$ is **(relatively) representable** if for each $X \in \mathscr{C}$ we have $F'' \stackrel{def}{=} F' \times_F h_X \cong h_{X'}$ for some $X' \in \mathscr{C}$.

Lemma 3.2.1. *Let $F \in \hat{\mathscr{C}}$. The **diagonal** $\Delta : F \to F \times F$ is representable if and only if each $f : X \to F$ is representable.*

Proof. If Δ is representable, then for any $Z \in \mathscr{C}$ we have an isomorphism of functors $\Delta' \stackrel{def}{=} F \times_\Delta h_Z \cong h_{Z'}$ for some $Z' \in \mathscr{C}$. So, if $f : X \to F$ then $X' \stackrel{def}{=} X \times_F Z'$ has $X \times_F h_{Z'} \cong h_{X'}$, i.e. the following commutes

$$\begin{array}{ccccc} X' & \longrightarrow & Z' & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & F & \xrightarrow{\Delta} & F \times F. \end{array}$$

Conversely, if each $f : X \to F$ is representable, then for any $Y \in \mathscr{C}$ there is an isomorphism of functors $F' \stackrel{def}{=} X \times_F h_Y \cong h_{Y'}$ for some $Y' \in \mathscr{C}$. Then for any $Z \to F \times F$ since $F \times_\Delta h_Z \cong h_Y$, i.e. the following commutes

$$\begin{array}{ccccc} Y' & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & F & \xrightarrow{\Delta} & F \times F, \end{array}$$

we see that Δ is representable. $\square$

Theorem 3.2.2. *(Yoneda's Lemma/The Fundamental Theorem of Category Theory) The functor of points* $\mathscr{C} \to \hat{\mathscr{C}}$ *is fully faithful, i.e. it induces isomorphisms of sets as follows. For each* $X, Y \in \mathscr{C}$ *there is a natural* $\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}(h_X, h_Y)$, *such that given* $T \xrightarrow{g} X \xrightarrow{f} Y$ *there is* $h_X \xrightarrow{f} h_Y$ *where* $h_X(T) \to h_Y(T)$ *is given by* $g \mapsto f \circ g$.

Thanks to this fact we are able to phrase much of our discussion of stacks explicitly and in particular we may discuss a fundamental phenomenon: descent, in familiar language. We conclude this section by stating a few definitions for our coming discussion of fibered categories and the analog of the sheaf condition for 2-categories. We say that a given category $\mathscr{C}$ is a **groupoid** if all of the maps in $\mathscr{C}$ are isomorphisms. Let $\mathscr{C}$ be a category. A **Grothendieck topology on** $\mathscr{C}$ is specified by the following data. For each $X \in \mathscr{C}$ there is a collection $\mathrm{Cov}(X)$, the coverings of X (containing $\{X_i \to X\}$) such that

1. if $V \to X$ is an isomorphism then $\{V \to X\} \in \mathrm{Cov}(X)$,
2. for each $\{X_i \to X\} \in \mathrm{Cov}(X)$ and for all $Y \to X \in \mathscr{C}$,
   (a) the fiber-product $X_i \times_X Y$ exists, and
   (b) $\{X_i \times_X Y \to Y\} \in \mathrm{Cov}(Y)$,
3. if $\{X_i \to X\} \in \mathrm{Cov}(X)$ and $\{V_{ij} \to X_i\} \in \mathrm{Cov}(X_i)$ then $\{V_{ij} \to X\} \in \mathrm{Cov}(X)$.

We say that the pair $(\mathscr{C}, \tau)$ for $\mathscr{C}$ a category and τ some Grothendieck topology on $\mathscr{C}$ is a **site**. However, for our reference later on while discussing rigid analytic geometry, we offer a more formal definition of a site.

Definition 3.2.3 ([Sta18c]). *A* ***site*** *is a pair of* $\mathscr{C}$ *a category and a set* $\mathrm{Cov}(\mathscr{C})$ *of families of morphisms with fixed target* $\{U_i \to U\}_{i\in I}$ *called* ***coverings of*** $\mathscr{C}$ *satisfying the axioms*

1. *If* $V \to U$ *is an isomorphism, then* $\{V \to U\} \in \mathrm{Cov}(\mathscr{C})$.

2. *If* $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ *and for each* i $\{V_{ij} \to U_i\}_{j\in J} \in \mathrm{Cov}(\mathscr{C})$, *then* $\{V_{ij} \to U\}_{i\in I, j\in J} \in \mathrm{Cov}(\mathscr{C})$.

3. *If* $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$ *and* $V \to U$ *is a morphism in* $\mathscr{C}$, *then* $U_i \times_U V$ *exists for all* i *and* $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathscr{C})$.

3.2.2 Descent and Fibered Categories

Let $\mathscr{C} \xrightarrow{\pi} \mathscr{D}$ be a functor. An arrow $X \to Y \in \mathscr{C}$ is **Cartesian** if for all $Z \in \mathscr{C}$ and all morphisms $\pi(Z) \to \pi(X)$ and $Z \to Y$, there exists a unique morphism $Z \to X$ such that the following commutes

$$\begin{tikzcd}
Z \arrow[dd, mapsto, "\forall_1"] \arrow[dr, dashed, "\exists_4 !"] \arrow[drr, bend left, "\forall_3"] & & \\
& X \arrow[r] \arrow[d] & Y \arrow[d] \\
\pi(Z) \arrow[r, "\forall_2"] & \pi(X) \arrow[r] & \pi(Y)
\end{tikzcd}$$

We call X a **pullback of** Y **along** $\pi(X) \to \pi(Y)$. We say $\mathscr{C} \xrightarrow{\pi} \mathscr{D}$ is a **fibered category** if for all $T' \xrightarrow{f} T \in \mathscr{D}$ and all $Y \in \mathscr{C}$ such that $Y \mapsto T$ there is some $X \in \mathscr{C}$ and $X \xrightarrow{g} Y \in \mathscr{C}$ such that g is Cartesian (i.e. $\pi(g) = f$). In other words, the following commutes

$$\begin{tikzcd}
\exists_3 X \arrow[r, "g"] \arrow[d, mapsto] & \forall_2 Y \in \mathscr{C} \arrow[d, mapsto] \\
\forall_1 T' \arrow[r, "f"] & T \in \mathscr{D}
\end{tikzcd}$$

and in this case we have

$$\begin{array}{l}
\mathscr{C} \ni \mathscr{C}(T) \to \mathscr{C}(T') \\
\downarrow \\
\mathscr{D} \ni T' \to T
\end{array}$$

where $\mathscr{C}(T) \to \mathscr{C}(T')$ is defined by $Y \mapsto X = \left(\begin{array}{c} \text{any choice of} \\ \text{Cartesian arrow} \end{array} \right)$.

Now we have a notion of the kind of categories (2-categories) from the first part of our informal definition of stacks as "fibered categories." We turn to the 2-categorical descent condition, the analog of the sheaf condition for 1-categories.

Suppose that $X' \xrightarrow{\rho} X$ is an étale cover in some site. That is, if $X = \bigcup_{i \in I} X_i$ is some open cover of X then $X' = \bigsqcup_{i \in I} X_i$. Let

$$\widetilde{X} = \widetilde{X}_{\mathrm{Zar}} \stackrel{def}{=} \left\{ \begin{array}{c} \text{sheaves of sets on } X \\ \text{in the Zariski topology} \end{array} \right\} / (\text{isomorphism of sheaves}).$$

Let $X'' = X' \times_X X' = \bigsqcup_{i,j \in I} X_{ij}$ and let $X''' = X' \times_X X' \times_X X' = \bigsqcup_{i,j,k \in I} X_{ijk}$. Then there is an equivalence of categories between the such fibered covers and sheaves:

$$
\begin{tikzcd}
\sqcup X_{ijk} \cong X''' \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & & \widetilde{X} \arrow[d, "\rho^{-1}"] \\
\sqcup X_{ij} \cong X'' \arrow[d, shift right=2, "\rho_2"] \arrow[d, shift left=2, "\rho_1"] & \overset{\cong}{\Leftrightarrow} & \widetilde{X'} \arrow[d, shift right=2, "\rho_2^{-1}"'] \arrow[d, shift left=2, "\rho_1^{-1}"] \\
\sqcup X_i \cong X' \arrow[d, "\rho"] & & \widetilde{X''} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
X & & \widetilde{X'''}
\end{tikzcd}
$$

Keeping our notation from above, we define the **category of descent data** $\widetilde{X' \xrightarrow{\rho} X}$ via the following data. The objects are pairs $(\mathscr{F}' \in \tilde{X}', i)$ where $i : \rho_1^{-1}\mathscr{F}' \xrightarrow{\sim} \rho_2^{-1}\mathscr{F}'$ are canonical isomorphisms such that given $\mathscr{F} \in \widetilde{X}$, if we write $\mathscr{F}_i \overset{def}{=} \mathscr{F}|_{X_i}$, the following commutes

Figure 3.1: The 2-categorical cocycle condition

$$
\begin{tikzcd}
& (\mathscr{F}_i|_{X_{ij}})|_{X_{ijk}} \arrow[r, "\sim", "\varphi_{ij}"'] & (\mathscr{F}_j|_{X_{ij}})|_{X_{ijk}} \arrow[dr, "\sim", "i"'] & \\
(\mathcal{F}_i|_{X_{ik}})|_{X_{ijk}} \arrow[ur, "\sim", "i"'] & & & (\mathcal{F}_j|_{X_{jk}})|_{X_{ijk}} \arrow[dl, "\sim", "\varphi_{jk}"'] \\
& (\mathscr{F}_k|_{X_{ki}})|_{X_{ijk}} \arrow[ul, "\sim", "\varphi_{ik}"'] & (\mathscr{F}_k|_{X_{jk}})|_{X_{ijk}} \arrow[l, "i", "\sim"'] &
\end{tikzcd}
$$

i.e. $\varphi_{jk}|_{X_{ijk}} \circ \varphi_{ij}|_{X_{ijk}} = \varphi_{ik}|_{ijk}$. The morphisms in the category of descent data are maps $(\mathscr{F}', i) \to (\mathscr{G}', \nu)$ with $\mathscr{F}' \xrightarrow{\psi} \mathscr{G}'$ such that the following commutes

$$\begin{array}{ccc} \rho_1^{-1}\mathscr{F}' & \xrightarrow{\rho_1^{-1}\psi} & \rho_1^{-1}\mathscr{G}' \\ \downarrow i & & \downarrow \nu \\ \rho_2^{-1}\mathscr{F}' & \xrightarrow{\rho_2^{-1}\psi} & \rho_2^{-1}\mathscr{G}' \end{array}$$

Now we can state formally the 2-categorical "sheaf" condition.

Theorem 3.2.4. *(Strong form of Descent)*
The functor $\widetilde{X} \to \widetilde{X' \xrightarrow{\rho} X}$ *defined by* $\mathscr{F} \mapsto (\rho^{-1}\mathscr{F}, i)$ *with* i *a canonical system of isomorphisms as above in Figure 3.1 and* ρ *a cover of* X *in some site, is an equivalence of categories.*

3.2.3 Defining a Stack

Finally we are able to move on to formal definitions of a stack and some special kinds of stacks which we will work with later in the document.

Definition 3.2.5. *Let* $(\mathscr{C}, \tau)$ *be a site. A* ***stack*** *over* $\mathscr{C}$ *is a category* $\mathscr{X} \to \mathscr{C}$ *fibered in groupoids satisfying descent, i.e. for each* $T' \xrightarrow{\rho} T \in \mathrm{Cov}(T)$ *there are morphisms*

$$\mathscr{X}(T) \xrightarrow{\rho^*} \mathscr{X}(T') \underset{\rho_1^*}{\overset{\rho_2^*}{\rightrightarrows}} \mathscr{X}(T'') \underset{\rho_{ij}^*}{\overset{}{\Rrightarrow}} \mathscr{X}(T'')$$

and descent implies there is an equivalence of categories $\mathscr{X}(T) \xrightarrow{\sim} \mathrm{Desc}(T' \to T)$ *where*

$$\mathrm{Desc}(T' \to T) \stackrel{def}{=} \varprojlim \left(\mathscr{X}(T') \rightrightarrows \mathscr{X}(T'') \Rrightarrow \mathscr{X}(T''') \right)$$

is the 2*-limit which contains pairs* $(x \in \mathscr{X}(T'), \sigma)$ *with* $\sigma : \rho_1^*(x) \xrightarrow{\sim} \rho_2^*(x)$ *satisfying the* 2*-cocycle condition 3.1.*

In particular we consider the following kinds of stacks exclusively.

Definition 3.2.6. *A stack* $\mathscr{X}$ *on a site* $(\mathscr{C}, \tau)$ *is* ***algebraic*** *or an Artin stack if*

1. *there exists some* $X \in \mathscr{C}$ *and a smooth cover* $X \to \mathscr{X}$, *and*

2. *the diagonal* $\mathscr{X} \xrightarrow{\Delta} \mathscr{X} \times \mathscr{X}$ *is representable.*

Another relevant hypothesis for computing canonical rings especially is the following.

Definition 3.2.7. *An algebraic stack* $\mathscr{X}$ *over a category* $\mathscr{C}$ *is a* ***Deligne-Mumford stack*** *if*

1. *there is some* $X \in \mathscr{C}$ *and an étale cover* $X \to \mathscr{X}$, *and*

2. *the diagonal* $\mathscr{X} \xrightarrow{\Delta} \mathscr{X} \times \mathscr{X}$ *is representable, quasi-compact and separated.*

In practice however, when working with stacks rather than proving a given moduli sapce is a stack for example, the following definition of a stack from [LRZ16] suffices for our work in this document.

Definition 3.2.8 ([LRZ16, Definition 2.1])**.** *A* ***stacky curve*** $\mathscr{X}$ *over an algebraically closed field* $\mathbb{K}$ *is a smooth proper integral scheme* $X/\mathbb{K}$ *of dimension* 1, *together with closed points* $P_1, \cdots, P_r$ *of* X *with stabilizer orders* $e_1, \cdots, e_r \in \mathbb{Z}_{\geqslant 2}$ *called* ***stacky points of*** $\mathscr{X}$.

Remark 3.2.9. *It is worth noting one crucial generalization which we have made in our assumptions when adopting the notion of a stacky curve from Definition 3.2.8 compared with the schemes of Chapter 2. Here we have made no restrictions on the characteristic of the ground field* $\mathbb{K}$ *and hence use different notation from Chapter 2 where the assumption of characteristic* 0 *is essential. Indeed, as we proceed to do, one can compute canonical rings and find Petri-style generators and relations for stacky curves even in positive characteristic.*

Suppose then that $\mathrm{char}(\mathbb{K}) = p$. *Let* X *be any smooth, projective curve over* $\mathbb{K}$. *Let* $G \leqslant \mathrm{Aut}(X)$ *be a finite group. Then the stack quotient* $[X/G]$ *has the structure of a stacky curve. Furthermore, if* $\gcd(\#G, \mathrm{char}(\mathbb{K})) = 1$ *then we say that* $[X/G]$ *is* ***tame*** *in the sense of [VZB22, Definition* 5.2.4*]; otherwise, we say* $\mathscr{X}$ *is* ***wild****. For the remainder of this work, our stacky curves in positive characteristic are tame. This is a necessary but not sufficient hypothesis for our positive-characteristic version of Petri-style calculations to behave similarly to the classical work in Chapter 2 and Appendices A and B in characteristic* 0. *See [VZB22, Chapter* 5*] for a more detailed treatement of this necessary but not sufficient condition.*

It is possible to work on wild stacky curves (see e.g. [VZB22, Remark 5.2.5*]), but beyond the scope of what we need now. Note that [VZB22, Remark* 5.2.5*] also explains the "similar behavior" we mention holds when considering canonical rings of tame stacky curves and schemes in characteristic* 0. *Finally, note that we discuss tameness of the stacks which are our main focus in Remark 6.2.9.*

3.3 How we use Stacks

See [Alp23] for a general stacks reference; see [VZB22] for an excellent and comprehensive reference on computing canonical rings of stacky curves and [O'D15] for a useful generalization of [VZB22] that we need for the Drinfeld setting. We are most interested in Deligne-Mumford stacks for this work, so some facts and examples will be specialized to that case, but we indicate when this occurs. We also discuss rigid analytic stacks and GAGA for rigid analytic and algebraic stacks, but leave that theory for a later section.

It is shown in e.g. [Lau96, Corollary 1.4.3] that the moduli space of rank r Drinfeld modules over the category of schemes of characteristic p is representable by a Deligne-Mumford algebraic stack of finite type over $\mathbb{F}_p$. One is able to compute the graded rings of global sections of line bundles on stacks which represent the Drinfeld moduli problems by means of geometric invariants with results that are slight variants on the theory in [VZB22]. We will follow [VZB22] in describing this computation, stating only select facts that we will need.

Recall from [VZB22, Definition 5.2.1], a **stacky curve** $\mathscr{X}$ over a field $\mathbb{K}$ is a smooth, proper, geometrically connected Deligne-Mumford stack of dimension 1 over $\mathbb{K}$ that contains a dense open subscheme. Every stacky curve $\mathscr{X}$ over a field $\mathbb{K}$ has a unique **coarse space morphism** $\pi : \mathscr{X} \to X$ with X a smooth proper integral scheme over $\mathbb{K}$ (called the coarse space) from Definition 3.2.8. Here π is universal for morphisms from $\mathscr{X}$ to schemes, and the set of isomorphism classes of F-points on $\mathscr{X}$ and X are in bijection for any algebraically closed field F containing $\mathbb{K}$. Note that étale locally on the coarse space X, a stacky curve $\mathscr{X}$ is the quotient of an affine scheme by a finite (constant) group $G \leqslant \mathrm{Aut}(X)$. For $x \in X$ some point, let G_x denote the stabilizer of x under the action by G. Only finitely many points of a stacky curve $\mathscr{X}$ have nontrivial stabilizers in the sense that a dense open subscheme of points all have isomorphic stabilizers, while some finitely many, the stacky points of Definition 3.2.8, have strictly larger stabilizer groups.

Continuing the notation in the last paragraph, let $\pi : \mathscr{X} \to X$ be a coarse space morphism. A **Weil divisor** is a finite formal sum of irreducible closed substacks of codimension 1 over $\mathbb{K}$. On a smooth Deligne-Mumford stack, every Weil divisor is Cartier. Any line bundle $\mathscr{L}$ on $\mathscr{X}$ is isomorphic to $\mathcal{O}_{\mathscr{X}}(D)$ for some Cartier divisor D. Finally, there is an isomorphism of sheaves on the Zariski site of X :

$$\mathcal{O}_X(\lfloor D \rfloor) \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathscr{X}}(D),$$

where

$$\lfloor D \rfloor = \Big\lfloor \sum_i a_i P_i \Big\rfloor \overset{def}{=} \sum_i \Big\lfloor \frac{a_i}{\# G_{P_i}} \Big\rfloor \pi(P_i).$$

Example 3.3.1. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of stacky curves with coarse spaces X and $Y = \operatorname{Spec} k$ for k some field respectively. Then the* ***sheaf of differentials*** *$\Omega^1_{\mathscr{X}} = \Omega^1_{\mathscr{X}/\operatorname{Spec} k}$ is the sheafification (see [Alp23, Section 2.2.9] for sheafification) of the presheaf on $\mathscr{X}_{\acute{e}t}$ given by*

$$(U \to \mathscr{X}) \mapsto \Omega^1_{\mathcal{O}_{\mathscr{X}}(U)/f^{-1}\mathcal{O}_{\mathscr{Y}}(U)},$$

where $\mathcal{O}_{\mathscr{X}}$ and $\mathcal{O}_{\mathscr{Y}}$ denote the structure sheaves on $\mathscr{X}$ and $\mathscr{Y}$ respectively (see e.g. [Alp23, Example 4.1.2] for more details on structure sheaves for Deligne-Mumford stacks).

Every smooth, projective curve X may be treated as a stacky curve with nothing stacky about it. On the other hand, as we have seen in Remark 3.2.9 the stack quotient $[X/G]$ for a finite group $G \leqslant \mathrm{Aut}(X)$ is a stacky curve, as in Definition 3.2.8. We know from e.g. [VZB22, Remark 5.2.8] that Zariski locally every stacky curve is the quotient of a smooth, affine curve by a finite group, so in some sense "most" stacky curves have a quotient description $[X/G]$ as above. Recall from [VZB22, Lemma 5.3.10.(b)] that the stabilizer groups of a tame stacky curve are isomorphic to the group of roots of unity μ_n for some n. In order to discuss Drinfeld moduli stacks, we introduce two more stacky notions.

We say a **gerbe** over a stacky curve is a smooth, proper, geometrically connected Deligne-Mumford stack of dimension 1 over its base field. Note that a gerbe is almost a stacky curve, except that it does not contain a dense open subscheme. Let $\mathscr{X}$ denote a geometrically integral Deligne-Mumford stack of relative dimension 1 over a base scheme S whose generic point has stabilizer μ_n for some n. Then there exists a stack, denoted $\mathscr{X} /\!\!/ \mu_n$, called the **rigidification** of $\mathscr{X}$, and a factorization

$$\mathscr{X} \xrightarrow{\pi} \mathscr{X} /\!\!/ \mu_n \to S$$

such that π is a μ_n-gerbe and the stabilizer of any point in $\mathscr{X} /\!\!/ \mu_n$ is the quotient of the stabilizer of the corresponding point in $\mathscr{X}$ by μ_n.

Remark 3.3.2. *In the factorization above, since π is a gerbe and furthermore is étale, the sheaf of relative differentials $\mathscr{X} \to \mathscr{X}/\!/\mu_n$ is 0, i.e. the gerbe does not affect sections of relative differentials (over the base scheme), nor the canonical ring which we define for stacks below. In particular, we can identify canonical divisors $K_{\mathscr{X}} \sim \pi^* K_{\mathscr{X}/\!/\mu_n}$, and the corresponding canonical rings are isomorphic.*

In particular, we treat seriously the stackiness of moduli spaces when we compute the following homogeneous coordinate rings on modular curves such as our consideration of Drinfeld modular curves.

Definition 3.3.3. *Let $\mathscr{X}$ be a stacky curve over a field k and let $\mathscr{L}$ be an invertble sheaf on $\mathscr{X}$. Then the* ***section ring*** *of $\mathscr{L}$ on $\mathscr{X}$ is the ring*

$$R(\mathscr{X}, \mathscr{L}) = \bigoplus_{d \geqslant 0} H^0(\mathscr{X}, \mathscr{L}^{\otimes d}).$$

If $\mathscr{L} \cong \mathcal{O}_{\mathscr{X}}(D)$ for some Cartier divisor D in particular, we can equivalently write

$$R_D = \bigoplus_{d \geqslant 0} H^0(\mathscr{X}, dD).$$

Recall from [VZB22, Chapter 5.1] that a **point** of a stack $\mathscr{X}$ is a map $\operatorname{Spec} F \to \mathscr{X}$ for F some field, and to a point x, we associate its stabilizer $G_x \stackrel{def}{=} \underline{\operatorname{Isom}}(x, x)$, a functor which is a representable by an algebraic space. If G_x is a finite group scheme, say that $\mathscr{X}$ is **tame** if $\deg G_x$ is not divisible by $\operatorname{char}(F)$ for any $x \in \mathscr{X}$. We say a point x with $G_x \neq \{1\}$ is a **stacky point** as in Definition 3.2.8.

Finally, for readability of our main results, we introduce some terminology inspired by [VZB22, Definition 5.6.2] and [VZB22, Proposition 5.5.6]. Let $\mathscr{X}$ be a tame stacky curve over an algebraically closed field $\mathbb{K}$ with coarse space X. A Weil divisor Δ on $\mathscr{X}$ is a **log divisor** if $\Delta = \sum_i P_i$ is an effective divisor given as a sum of distinct points (stacky or otherwise) on $\mathscr{X}$. By [VZB22, Proposition 5.5.6] if $K_{\mathscr{X}}$ and K_X are canonical divisors on $\mathscr{X}$ and X respectively, then there is a linear equivalence of divisors

$$K_{\mathscr{X}} \sim K_X + R = K_X + \sum_x \left(1 - \frac{1}{\deg G_x}\right) x,$$

where G_x is the stabilizer of a closed substack $x \in \mathscr{X}$, and the sum above is taken over closed substacks of $\mathscr{X}$.

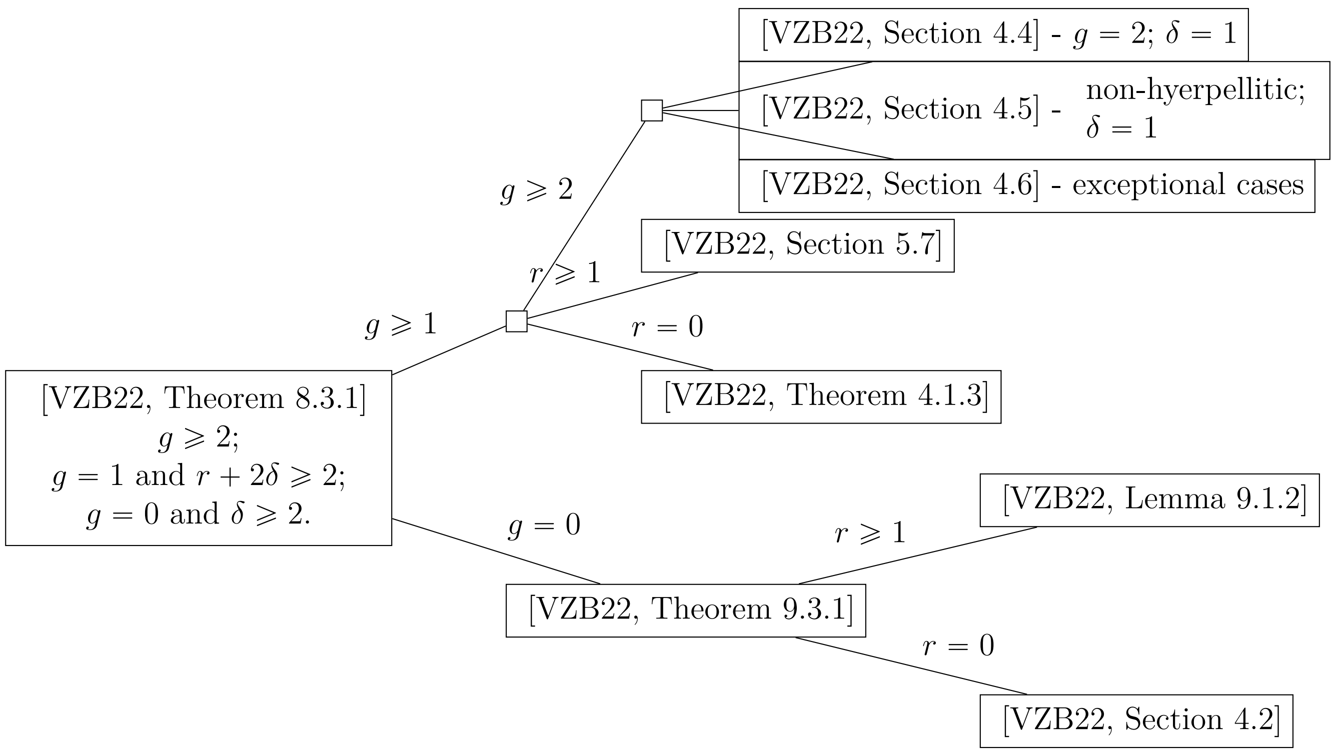
Our main object of interest is defined in [VZB22, Defintions 5.6.1 and 5.6.2] which we generalize slightly to allow for stacky points in a log divisor: the canonical ring of a log stacky curve is the ring

$$R_D = \bigoplus_{d \geqslant 0} H^0(\mathscr{X}, dD),$$

where $D = K_{\mathscr{X}} + \Delta$, for Δ a log divisor on $\mathscr{X}$.

Recall from [VZB22, Definition 5.6.6] that the **signature** of a log stacky curve $(\mathscr{X}, \Delta)$ is the tuple $(g; e_1, \ldots, e_r; \delta)$ where g is the genus of the coarse space X, the integers $e_1, \ldots, e_r$ are the orders of the stabilizers of geometric points of $\mathscr{X}$ with non-trivial stabilizers ordered such that $e_i \leqslant e_{i+1}$ for all i, and $\delta = \deg \Delta$. The main results of [VZB22] are organized around their inductive Theorem [VZB22, 8.3.1] which succesively computes $R(\mathscr{X}, \Delta)$ for $(\mathscr{X}, \Delta)$ with signature $(g; e_1, \ldots, e_r; \delta)$ in terms of canonical rings of log stacky curves $(\mathscr{X}', \Delta)$ with signature $(g; e_1, \ldots, e_{r-1}; \delta)$. We summarize the way that their result splits into various base cases with the following figure.

Figure 3.2: A Map of the Inductive Result in [VZB22]



We also note some generalizations of [VZB22] to section rings of $\mathbb{Q}$-divisors. Such divisors are, as in [VZB22, Remark 5.6.4], often useful for log canonical rings in more pathological situations than our current context, such as the case of "wild" ramification of stacky points, for example. See [O'D15] for general $\mathbb{Q}$-divisors on genus 0 curves, see [CFO24] for $\mathbb{Q}$-divisors on elliptic curves, and thanks to [LRZ16] one is able to at least tightly bound the degrees of generators and relations for spin canonical rings of log stacky curves in all genera.

We remark, as in the Introduction to [CFO24], that [CFO24] more or less concludes the line of inquiry in computing explicit minimal presentations of general section rings. The results of both [O'D15] and [CFO24] are sufficiently complicated and combinatorial in nature so as to appear to John Voight "as much some kind of additive number theory as algebraic geometry," and in neither work is the notion of a stacky curve as such relevant for proofs. Somehow it is too arbitrary to ask for general $\mathbb{Q}$-divisors, especially since for divisors of low degree or ineffective divisors, even on elliptic curves, the section ring has a rather complicated presentation. There is no reason to believe that for curves in higher genus, where ampleness of divisors requries greater degree, that a description of such section rings will have any kind of uniform principle to it. Furthermore, this is a rather algorithmic problem, where Magma and the existing theory is enough to bootstrap some kind of presentation for a section ring in a given example, whereas it is quite challenging and likely not aesthetically interesting to describe some general theory. Finally, it should not be dismissed how high of a bar is set by [VZB22]. This work covers curves in all genera with great detail and is more often than not sufficient for the number-theory motivated calculations we are concerned with in this thesis.

Chapter 4

Drinfeld Setting

In this chapter we discuss the arithmetic of function fields and introduce the Drinfeld setting. Because of the well-established analogy between number fields and function fields many results from class field theory such as Kronecker-Weber have a corresponding theorem for function fields. However, as we are interested in arithmetic geometry more than class field theory in particular, our description of this analogy will be focused more on Drinfeld modules - the analogs of abelian varieties over a number field, and their moduli. As we will later see, certain moduli spaces of Drinfeld modules with level structure behave quite like moduli of elliptic curves, which provides us a template for our theory of Drinfeld modular curves. This is foundational material for our main results which describe the geometry of Drinfeld modular forms in a manner quite like the more familiar case of modular forms over $\mathbb{C}$ or number fields.

4.1 Notation and the "Setting"

References for Drinfeld modular curves are [Gek86], [Gek01] and [MS15]; for Drinfeld modular forms see the survey [Gek99] and the papers [GR96], [Gek88], [Bre16], [Cor97a] and [DK23]. For the theory of Drinfeld modules themselves the best reference is [Pap23]. Before we discuss these objects, we quickly recall some basics of function fields.

Let $\mathbb{F}_q$ be the finite field of order q a power of an odd prime. As function-field analogs of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ define the rings $A = \mathbb{F}_q[T]$, $K = \mathrm{Frac}(A) = \mathbb{F}_q(T)$, $K_\infty = \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$, the completion of K at the place ∞, and let $C = \widehat{\overline{K_\infty}}$ be the completion of the algebraic closure of K_∞ respectively. Then C is an algebraically closed, complete, and non-archimedean field.

We might just as well have taken K to be the function field of any smooth, connected, projective curve over a field of characteristic q rather than our particular choice of K as the function field of $\mathbb{P}^1$. Our specification of this function field in particular is only for ease with notation.

We have the usual discrete valuation $v : K^\times \to \mathbb{Z}$ given by

$$v\left(\frac{\sum_1^n a_i T^i}{\sum_1^m b_i T^i}\right) = m - n$$

which we extend to the Laurent series K_∞ by

$$v\left(\sum_{i \geqslant n} a_i T^i\right) = -n \quad \text{and} \quad v(0) = \infty.$$

The corresponding metric, which we extend to C, is the non-archimedean norm defined by $|f| = q^{-v(f)}$.

The Drinfeld-setting version of the upper half-plane $\mathcal{H} \subset \mathbb{C}$ is $\Omega \overset{def}{=} C - K_\infty$. We will discuss this is more detail in the next section.

Note that the group $\mathrm{GL}_2(A)$ acts on Ω by Möbius transformations as SL_2 acts on $\mathcal{H}$, but $\det(\gamma) \in \mathbb{F}_q^\times$ for $\gamma \in \mathrm{GL}_2(A)$. Let $N \in A$ be a non-constant, monic polynomial and let $\Gamma(N)$ be the subgroup of $\mathrm{GL}_2(A)$ with matrices congruent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo N. A subgroup Γ of $\mathrm{GL}_2(A)$ such that $\Gamma(N) \subseteq \Gamma$ for some N is a **congruence subgroup** and we call such an N of the least degree the **conductor** of Γ.

Some important examples of congruence subgroups are the following:

$$\Gamma_1(N) = \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \pmod{N}\right\} \text{ and } \Gamma_0(N) = \left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{N}\right\}.$$

We establish an important assumption for this work: throughout, $\Gamma \leqslant \mathrm{GL}_2(A)$ is some congruence subgroup such that for every $\alpha, \alpha' \in \mathbb{F}_q^\times$, Γ contains the matrices of form $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha' \end{smallmatrix}\right)$, that is, the diagonal matrices in $\mathrm{GL}_2(A)$. This means we have $\det \Gamma = \{\det(\gamma) : \gamma \in \Gamma\} = \mathbb{F}_q^\times$. In general $\det \Gamma$ is a subgroup of $\mathbb{F}_q^\times$.

Let $(\det \Gamma)^2$ be the set of squares of elements in $\det \Gamma : (\det \Gamma)^2 = \{x^2 : x \in \det \Gamma\}$. Let

$$\Gamma_2 \stackrel{def}{=} \{\gamma \in \Gamma : \det(\gamma) \in (\det \Gamma)^2\}.$$

When we write $\Gamma \leqslant \mathrm{GL}_2(A)$, we mean Γ satisfying the conditions above, so $\det \Gamma_2 = (\mathbb{F}_q^\times)^2$.

The condition that Γ has all possible determinants is simply for ease of notation, as it is more pleasant to compute congruences modulo $q-1$ rather than $\# \det \Gamma$. Our emphasis on the case when q is odd is essential as we make repeated use of the fact that $q-1$ is even.

We will make use of a kind of "parity" for congruence subgroups for which we introduce the following terminology:

Definition 4.1.1. *We say that a congruence subgroup Γ is **square** if there is some $z \in \Omega$ such that the stabilizer $\Gamma_z = \{\gamma \in \Gamma : \gamma z = z\}$ strictly contains $\mathbb{F}_q^\times \cong \left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) : \alpha \in \mathbb{F}_q^\times\right\}$ and every $\gamma \in \Gamma_z \backslash \mathbb{F}_q^\times$ has $\det \gamma$ a square in $\mathbb{F}_q^\times$. Likewise, Γ is **non-square** if it contains a stabilizer Γ_z for some $z \in \Omega$ strictly larger than $\mathbb{F}_q^\times$ and some γ with $\det \gamma \in \mathbb{F}_q^\times \backslash (\mathbb{F}_q^\times)^2$.*

In our application stabilizers are all $\mathrm{GL}_2(A)$-conjugate subgroups of $\mathbb{F}_{q^2}^\times$ so that one only needs to check for a single point $z \in \Omega$ with a stabilizer $\Gamma_z \supsetneq \mathbb{F}_q^\times$ whether Γ_z contains some matrix with a non-square determinant.

4.2 The Burhat-Tits Tree

We can describe a fundamental domain for the Drinfeld "upper half plane" Ω following [Gek86, Chapter V.1]. Thanks to Tristan Phillips we can even include a cartoon. Along the way we introduce an important tool used to study the geometry of the Drinfeld setting that we return to in Section 8.3.

Let $\mathscr{T}$ be the **Bruhat-Tits** tree of $\mathrm{PGL}_2(K_\infty)$ defined in [Ser80, Section 2.1]. Though we do not offer an exhaustive definition, we discuss many properties of $\mathscr{T}$. The Burhat-Tits tree is a connected tree, i.e. a simply connected simplicial complex. The vertices of $\mathscr{T}$ are similarity classes of $\mathcal{O}_\infty$-lattices in K_∞, where $\mathcal{O}_\infty$ is the ring of integers in K_∞. We say two vertices $L_1 \neq L_2$ are **adjacent** if we can choose L_1 and L_2 in their similiarity classes such that $L_1 \subset L_2$ of index q^δ where $\delta = \deg(\infty)$ is the degree of the residue field of K at ∞ over $\mathbb{F}_q$. Each vertex has $q^\delta + 1$ neighbors. We can assign a metric $d(x, y)$ to the realization $\mathscr{T}(\mathbb{R})$ which gives distance 1 to adjacent vertices and is linear on edges.

Consider a building map $\lambda : \Omega \to \mathscr{T}(\mathbb{R})$ given by $z \mapsto$ similarity class of $|\ |_z$, where for $(x, y) \in K_\infty^2$ we say

$$|(x, y)|_z = |zx + y|.$$

We can give a topological picture of the "buidling" which λ does with the construction:

- for each vertex of $\mathscr{T}$, take a copy of

$$\mathbb{P}^1(C) - \left(\begin{array}{l} \text{a union of } q^\delta + 1 \text{ open balls} \\ \text{with disjoint closures} \end{array} \right)$$

  and

- for each edge of $\mathscr{T}$, take an annulus $\mathbb{P}^1(C) -$ (a union of 2 disjoint open balls),
- then glue these according to incidence in $\mathscr{T}$.

The result is a 2-dimensional manifold which is the boundary of a tubular neighborhood of $\mathscr{T}(\mathbb{R})$. Then λ is no more than a projection onto $\mathscr{T}(\mathbb{R})$. When $q = 2$, $\deg(\infty) = 1$ so $q^\delta + 1 = 3$, and Tristan Phillips sketched (by hand) the following cartoon:

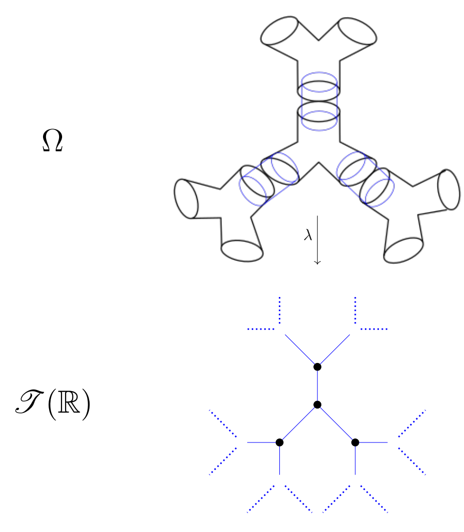


Figure 4.1: The Bruhat-Tits tree $\mathscr{T}(\mathbb{R})$ and a Fundamental Domain for the Drinfeld "upper half-plane" Ω

4.3 Drinfeld modules

The theory of Drinfeld modules is rich in both algebraic and analytic structure. Both interpretations and their equivalence are important in understanding the moduli spaces of Drinfeld modules of a given rank. We state only what we need for our computation of the canonical ring of certain log-stacky moduli spaces and the corresponding algebras of Drinfeld modular forms. A concise and accessible introduction to this material is the article [Poo22] and much more detail is covered in [Pap23].

Analytic Approach

We give a quick description of Drinfeld modules as lattice quotients. Following Breuer [Bre16], we say an A-submodule of C of form $\Lambda = \omega_1 A + \cdots + \omega_r A$, for $\omega_1, \cdots, \omega_r \in C$ some K_∞-linearly independent elements, is an **A-lattice of rank** r. Then we define an exponential function as follows.

Definition 4.3.1. *Let $\Lambda \subset C$ be an A-lattice of rank r. The* ***exponential function of*** Λ*, denoted $e_\Lambda : C \to C$, is defined by*

$$e_\Lambda(z) \stackrel{def}{=} z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right).$$

For any A-lattice, the exponential e_Λ is holomorphic in the rigid analytic sense (see e.g. [FvdP04, Definition 2.2.1]), surjective, $\mathbb{F}_q$-linear, Λ-periodic and has simple zeros on Λ. By an $\mathbb{F}_q$-linear function we mean the following.

Lemma 4.3.2. *Let $\mathbb{K}$ be a field of characteristic p containing $\mathbb{F}_q$. Then $f(x) \in \mathbb{K}[x]$ is $\mathbb{F}_q$**-linear** (i.e. $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{F}_q$) if and only if $f(x) = \sum_{i=0}^{n} a_i x^{q^i}$.*

Let $C\{X^q\} \overset{def}{=} \{a_0X + a_1X^q + \cdots + a_nX^{q^n} : a_0, \cdots, a_n \in C,\ n \geqslant 0\}$ denote the non-commutative polynomial ring of $\mathbb{F}_q$-linear polynomials over C, with the operation of multiplication given by composition. Note we can use any A-algebra B in place of C to define a similar polynomial ring $B\{X^q\}$ to $C\{X^q\}$. For each $a \in A$ the exponential satisfies the functional equation

$$e_\Lambda(az) = \varphi_a^\Lambda(e_\Lambda(z)),$$

where $\varphi_a^\Lambda(X) \in C\{X^q\}$ is some element of degree $q^{r \deg a}$. Then we say a ring homomorphism $\varphi : A \to C\{X^q\}$ given by

$$a \mapsto \varphi_a^\Lambda \overset{def}{=} a_0(a)X + \cdots + a_{r \deg a}(a)X^{q^{r \deg a}},$$

(an $\mathbb{F}_q$-algebra monomorphism) is a **Drinfeld module of rank** r if the coefficient with largest index is non-zero.

Algebraic Approach

We recall, without any proofs, some facts concerning the algebraic theory which corresponds to the definition above. A more complete discussion of these next facts is found in [Pap23, Definition 3.1.4] and [Pap23, Lemma 3.1.4]. We are mostly interested in the notation.

We state the following result so that when we define a moduli space of Drinfeld modules, we can make sense of Drinfeld modules over an arbitrary base scheme and therefore eventually have a well-defined category fibered in groupoids when we consider moduli stacks later.

Theorem 4.3.3 ([Wat79, Page 65])**.** *Let B be an A-algebra, and let $\mathbb{G}_{a,B}$ denote the affine additive group scheme over B represented by* $\operatorname{Spec} B[t]$*. Then the set* $\operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a,B})$ *of $\mathbb{F}_q$-linear endomorphisms of $\mathbb{G}_{a,B}$, is* $\operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_{a,B}) \cong B\{X^q\}$.

Proof. Let $A = k\langle x\rangle$. Then A represents the group scheme $\mathbb{G}_a$, i.e. there is a ring homorphism $A \to \operatorname{End}_{\mathbb{F}_q}(\mathbb{G}_a)$ since affine group schemes correspond to Höpf algebras. Suppose that $g, h : A \to R$ are A-algebra maps with $g(x) = r$ and $h(x) = s$. Then we need a $\Delta : A \to A \otimes A$ such that the composite $(g, h) \circ \Delta : A \to A \otimes A \to R$ sends $x \mapsto r + s$. The map Δ given by $x \mapsto x \otimes 1 + 1 \otimes x$ does this and is the unique map we want since the Yoneda correspondence is a bijection.

Endomorphisms of $\mathbb{G}_a$ correspond to $Q(x) \in k\langle x\rangle$ with $\Delta Q = Q \otimes 1 + 1 \otimes Q(x)$. In particular, if $Q(x) = \sum a_r x^r$ then $a_r(x\otimes 1+1\otimes x)^r = a_r(x^r\otimes 1+1\otimes x^r)$ so $a_0 = 0$. If $r = p^n s$ for some $s > 1$ which is coprime to p, then $(x \otimes 1 + 1 \otimes x)^r = (x^{p^n} \otimes 1 + 1 \otimes x^{p^n})^s$ has a term $s(x^{p^n} \otimes x^{(s-1)p^n})$ and $a_r = 0$. Then $Q(x) = \sum b_j x^{p^j}$.

Then since $Q(x) = x^p$ corresponds to $\tau(x) = x^p \in \operatorname{End}(\mathbb{G}_a)$, the composite map scaling by some b after τ^n gives a map $x \mapsto bx^{p^n}$. Therefore any $\varphi \in \operatorname{End}(\mathbb{G}_a)$ is uniquely expressed by some $\sum b_i x^{q^i}$. We also have $x^q b = b^p x^q$. □

Finally, we can introduce algebraic Drinfeld modules over any scheme.

Definition 4.3.4. *A **Drinfeld module** of rank r over an A-scheme S is a pair (E, φ) consisting of:*

- *a $\mathbb{G}_a$-bundle E (e.g. an additive group scheme) over S such that for all $U = \operatorname{Spec} B$ an affine open subset of S for B an A-algebra in the Zariski topology on S, there is an isomorphism $\psi : E|_U \xrightarrow{\sim} \mathbb{G}_{a,B}$ of group schemes over U*

- *a ring homomorphism $\varphi : A \to \operatorname{End}(E)$*

such that for a family of pairs (U_i, ψ_i) which form a trivializing cover of E (i.e. $U_i = \operatorname{Spec} B_i$ are an affine open cover and $\psi_i : E_{\pi^{-1}(U_i)} \xrightarrow{\sim} \mathbb{G}_{a,B_i}$ are local isomorphisms of additive group schemes), the morphism φ restricts to give maps $\varphi_i : A \to \operatorname{End}(\mathbb{G}_{a,B_i})$ of form $\varphi_i(T) = TX + b_{1,i}X^q + \cdots + b_{r,i}X^{q^r}$, compatible with the transition functions $\psi_{ji} = \psi_i \circ \psi_j^{-1}$, i.e. $\varphi_j \circ \psi_{ij} = \psi_{ij} \circ \varphi_i$ on all intersections $U_{ij} = U_i \cap U_j$.

Remark 4.3.5. *In the special case when we consider Drinfeld modules over a field, the algebraic definition of a Drinfeld module is simpler. In particular, we have $E = \mathbb{G}_a$, and we do not need any of the trivializations of our bundle as we are working over a single affine scheme. Therefore, it suffices to provide a ring homomorphism $\varphi : A \to \operatorname{End}(\mathbb{G}_a)$. We do not make further explicit use of the algebraic definition of Drinfeld modules in this article beyond the following examples.*

Recall from [Pap23, Definition 3.3.1] that a **morphism of Drinfeld modules** $u : \varphi \to \psi$ over a field $\mathbb{K}$ of characteristic p is some polynomial $u \in \mathbb{K}\{X^q\}$ such that $u\varphi_a = \psi_a u$ for all $a \in A$, where X is an indeterminant. A non-zero morphism $u : \varphi \to \psi$ is called an **isogeny**, and we define the group

$$\operatorname{End}_{\mathbb{K}}(\varphi) \overset{def}{=} \operatorname{Hom}_{\mathbb{K}}(\varphi, \varphi).$$

Under composition $\operatorname{End}_{\mathbb{K}}(\varphi)$ is a subring of $\mathbb{K}\{X^q\}$ which we call the **endomorphism**

ring of φ. The **automorphisms** of a Drinfeld module φ are the invertible elements of its endomorphism ring.

The **determinant** of a rank 2 Drinfeld module $\varphi_2^z(T) = TX + gX^q + \Delta X^{q^2}$ is the rank 1 Drinfeld module

$$\psi^z(T) \overset{def}{=} TX - \Delta X^q.$$

Example 4.3.6 ([Car38])**.** *The **Carlitz module** is the rank* 1 *Drinfeld module defined by*

$$\varphi(T) = TX + X^q,$$

and corresponds to the lattice $\overline{\pi}A \subset \Omega$. *Here,* $\overline{\pi} \in K_\infty(\sqrt[q-1]{-T})$ *is the **Carlitz period**, defined up to a* $(q-1)$*st root of unity. We fix one such* $\overline{\pi}$ *once and for all.*

As an algebraic Drinfeld module, the Carlitz module is the image of the ring homomorphism

$$\begin{aligned} \varphi :& A \to C\{X^q\} \\ & T \mapsto TX + X^q \end{aligned}$$

which is a rank 1 *module since* $\deg \varphi = q = |T|^1$, *over the* A*-scheme* $\operatorname{Spec} C$. *Here,* $|\cdot|$ *is the extension of the* ∞*-adic absolute value to* C.

Example 4.3.7. *Let* $z \in \Omega$, *and consider the rank* 2 *lattice* $\Lambda_z = \overline{\pi}(zA + A)$. *The associated Drinfeld module of rank* 2 *is*

$$\varphi^z(T) = TX + g(z)X^q + \Delta(z)X^{q^2},$$

where g and Δ are Drinfeld modular forms of type 0 *and weights* $q-1$ *and* q^2-1 *respectively. We will define Drinfeld modular forms in the next section. This is analogous to defining an elliptic curve by a short Weierstrass equation whose coefficients are values of Eisenstein series. Once again this is an algebraic Drinfeld module over an affine A-scheme. We have written down in particular the image of a degree* 2 *ring homomorphism $\varphi : A \to C\{X^q\}$. The Carlitz period $\overline{\pi}$ serves to normalize the coefficients of the series expansion of g and Δ at the cusps of* $\mathrm{GL}_2(A)$ *so that those coefficients are elements of A.*

For our intuition, we offer some further descriptions of algebraic Drinfeld modules. Globally, the data of a Drinfeld module is a pair (φ, G) with $\varphi : A \to \mathrm{End}(G)$. Locally, the data of a Drinfeld module is

- a family of pairs (U_i, ψ_i), where the $U_i = \mathrm{Spec}(B_i)$ are affine opens which form a trivializing cover for G, i.e. we have isomorphisms of group schemes $\psi_i : G|_{\pi^{-1}(U_i)} \xrightarrow{\sim} \mathbb{G}_{a,B_i}$ and "restricted" Drinfeld modules for the Zariski topology;

- some ring homomorphisms $\varphi_i : A \to \mathrm{End}(\mathbb{G}_{a,B_i})$ such that $(\mathbb{G}_{a,B_i}, \varphi_i)$ are "restricted" Drinfeld modules,

such that $\psi_{ji} = \psi_i \circ \psi_j^{-1}$ is an isomorphism of "restricted" Drinfeld modules on intersections of affines. In other words, $\mathbb{G}_a$-bundles are automatically line bundles. We sketch the argument as follows.

Consider the diagram

$$\begin{tikzcd}
B_{ij}[x] \arrow[r, "\psi_{ij}^{\#}"] \arrow[d, "\varphi_i^{\#}(T)"] & B_{ij}[x] \arrow[d, "\varphi_j^{\#}(T)"] \\
B_{ij}[x] \arrow[r, "\psi_{ij}^{\#}"] & B_{ij}[x]
\end{tikzcd}$$

We want to show that the transition maps $\psi_{ij}^{\#} : B_{ij}[x] \to B_{ij}[x]$ are forced to have form $x \mapsto m_{ij}x$ for some $m_{ij} \in B_{ij}^{\times}$. We chase x around the diagram and ensure that the endomorphisms φ_i descend, i.e. satisfy the cocyle condition. That is, our endomorphisms respect or commute with the transition maps ψ_{ij}. Note that we are working on the corresponding algebras rather than directly on the group schemes, that is on the functor of points, because this makes the computations algebraic and familiar. Recall that we have $x \overset{\psi_{ij}^{\#}}{\mapsto} m_{ij}x$ and

$$x \overset{\varphi_i^{\#}}{\mapsto} Tx + b_{1,i}x^q + b_{2,i}x^{q^2} + \cdots + b_{d,i}x^{q^d}.$$

So, we compute

$$\begin{aligned}
m_{ij}^{-1}\varphi_j^{\#}(T)m_{ij} &= m_{ij}^{-1}(Tx + b_{1,j}x^q + b_{2,j}x^{q^2} + \cdots + b_{d,j}x^{q^d})m_{ij} \\
&= Tx + m_{ij}^{q-1}b_{1,j}x^q + \cdots + m_{ij}^{q^d-1}b_{d,j}x^{q^d} \\
&= Tx + b_{1,i}x^q + b_{2,i}x^{q^2} + \cdots + b_{d,i}x^{q^d} \\
&= \varphi_i^{\#}(T),
\end{aligned}$$

or in other words $m_{ij}\varphi_i(T) = \varphi_j(T)m_{ij}$.

Example 4.3.8. *An automorphism of the rank 2 Drinfeld module $\varphi_T = TX + gX^q +$*

ΔX^{q^2} *over* C *where* $g \neq 0$ *is given by* $\alpha^{-1}\varphi_T\alpha$ *for some* $\alpha \in C^\times$. *We have*

$$\begin{aligned} \alpha^{-1}\varphi_T\alpha &= \alpha^{-1}\left(TX + gX^q + \Delta X^{q^2}\right)\alpha \\ &= TX + \alpha^{q-1}gX^q + \alpha^{q^2-1}\Delta X^{q^2} \\ &= \varphi_T, \qquad\qquad \text{\textit{if} } \alpha \in \mathbb{F}_q^\times. \end{aligned}$$

Chapter 5

Rigid Analytic Stacks; Rigid Stacky GAGA

We fix once and for all some $\overline{\pi} \in K_\infty(\sqrt[q-1]{-T})$, a **Carlitz period**, defined up to a $(q-1)$st root of unity. We define a **parameter at infinity**

$$u(z) \stackrel{def}{=} \frac{1}{e_{\overline{\pi}A}(\overline{\pi}z)} = \frac{1}{\overline{\pi}e_A(z)} = \overline{\pi}^{-1}\sum_{a\in A}\frac{1}{z+a}.$$

We discuss the parameter u in more detail in coming chapters.

If we are discussing a stack, we may sometimes write "DM" as shorthand for "Deligne-Mumford." There should be no confusion with the phrase Drinfeld module in particular as we will not be thinking about Drinfeld modules explicitly.

5.1 Rigid Analytic Spaces

We briefly recall some definitions we need to discuss rigid anaytic spaces, which are the natural means to discuss quotients of the Drinfeld "upper half-plane" Ω by congruence subgroups. A more thorough treatment and good reference for rigid analytic geometry in general is [FvdP04]. We will specialize to rigid analytic spaces over C for readability.

We need the following two intermediate definitions to define a rigid analytic space.

Definition 5.1.1 ([FvdP04, Page 46]). *Let $z_1, \cdots, z_n$ denote some variables. Let $T_n = C\langle z_1, \cdots, z_n\rangle$ be the n-dimensional C-algebra which is the subring of the ring of formal power series $C[[z_1, \cdots, z_n]]$*

$$T_n = \left\{ \sum_{\alpha} c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} \in C[[z_1, \cdots, z_n]] : \lim c_\alpha = 0 \right\},$$

*where $\alpha = (\alpha_1, \cdots, \alpha_n)$. An **affinoid algebra** A over C is a C-algebra which is a finite extension of T_n for some $n \geqslant 0$.*

Definition 5.1.2 ([FvdP04, Definition 2.4.1]). *Let X be a set. A **G-topology** on X consists of the data:*

1. *a family $\mathscr{F}$ of subsets of X such that $\varnothing, X \in \mathscr{F}$ and if $U, V \in \mathscr{F}$, then $U \cap V \in \mathscr{F}$; &*

2. *for each $U \in \mathscr{F}$, a set $\mathrm{Cov}(U)$ of coverings of U by elements of $\mathscr{F}$ (We say the $U \in \mathscr{F}$ are **admissible sets** and the elements of $\mathrm{Cov}(U)$ are **admissible***

coverings*);*

such that admissible coverings satisfy the axioms of a site Definition 3.2.3.

To recall the site axioms, we say that in the site specified by this “admissible topology” the following conditions are met:

- $\{U\} \in \mathrm{Cov}(U)$;
- for each $U, V \in \mathscr{F}$ with $V \subset U$ and $U \in \mathrm{Cov}(U)$, the covering $U \cap V \overset{def}{=} \{U' \cap V : U' \in U\}$ belongs to $\mathrm{Cov}(U)$;
- let $U \in \mathscr{F}$, let $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ and let $\mathcal{U}_i \in \mathrm{Cov}(U_i)$. Then

  $$\bigcup_{i\in I} \mathcal{U}_i \overset{def}{=} \{U' : U' \text{ belongs to some } \mathcal{U}_i\}$$

  is an element of $\mathrm{Cov}(U)$.

Remark 5.1.3. *G-topology is an abbreviation of Groethendieck topology, so we do not type the “G” in math-mode.*

Example 5.1.4 ([FvdP04, Definition 4.2.1](Weak G-topology))**.** *Let $X = \mathrm{Sp}(A)$ be an affinoid space over C, i.e. A is a C-affinoid algebra. Any surjective map of C-affinoid algebras $T_n \to A$ induces an embedding of X in the standard polydisk $\mathrm{Sp}(T_n)$. The topology on C induces a topology on this polydisk $\mathrm{Sp}(T_n)$ and so on X as well, and this topology is canonical in the sense that it does not depend on the choice of embedding.*

*The admissible subsets of X for the **very weak** G-topology on X are the rational subsets (see [FvdP04, Definition 4.1.1]) - $R \subset X = \mathrm{Sp}(A)$ is **rational** if there exists $f_0, \cdots, f_s \in A$ generating the unit ideal in A such that*

$$R = \{x \in X : |f_i(x)| \leqslant |f_0(x)| \text{ for } i = 1, \cdots, s\}.$$

A covering $\{R_i\}_{i\in I}$ of a rational subset R by rational subsets R_i is admissible for the very weak G-topology of there exists a finite $J \subset I$ with $R = \cup_{i\in J} R_i$.

*The **weak** G-topology on X consists of the admissible sets which are finite unions of rational subsets, and an admissible covering $\{R_i\}_{i\in I}$ of an admissible R has the features that: all R_i are admissible and there exists a finite $J \subset I$ with $R = \cup_{i\in J} R_i$. The weak topology is slightly finer than the very weak.*

Finally, we come to the point:

Definition 5.1.5 ([FvdP04, Definition 4.3.1])**.** *A **rigid analytic space** is a triple $(X, T_X, \mathcal{O}_X)$ consisting of a set X, a G-topology T_X on X and a structure sheaf of C-algebras $\mathcal{O}_X$ on X for which there exists an admissible open covering $\{X_i\}$ of X such that each $(X_i, T_{X_i}, \mathcal{O}_{X_i})$ is an affinoid over C and $U \subset X$ belongs to T_X if and only if $U \cap X_i$ belongs to T_X for each i.*

Example 5.1.6. *Consider the Drinfeld "upper half-plane" $\Omega = \mathbb{P}^1(C) - \mathbb{P}^1(K_\infty)$. We know $\mathbb{P}^1(K_\infty)$ is compact in the rigid analytic topology, so we know from [GR96, Section 1.2] that Ω is a rigid analytic space. The action by $\Gamma \leqslant \mathrm{GL}_2(A)$ a congruence subgroup on Ω by Möbius transformations has finite stabilizer for each $z \in \Omega$, and as in [GR96, Sections (2.5) and (2.6)], $\Gamma\backslash\Omega$ is a rigid analytic space.*

Now we will follow [Vin12, Section 3.1.1.] for an explicit admissible open (pure) cover of Ω:
We have a valuation $v_\infty(x) \overset{def}{=} -\deg(x)$ for K with local parameter T, which gives rise to an absolute value normalized so that $|x| = q^{\deg(x)}$. For $n \in \mathbb{Z}$ let

$$D_n \overset{def}{=} \left\{ z \in C : \begin{array}{c} q^{-n-1} \leqslant |z| \leqslant q^{-n}, \text{ and} \\ |z - cT^{-n}| \geqslant q^{-n}, |z - cT^{-n-1}| \geqslant q^{-n-1} \text{ for all } c \in \mathbb{F}_q^\times \end{array} \right\}.$$

Then $D_n \subset \Omega$ is an affinoid space over K_∞. For $x \in K_\infty$, let $D_{(n,x)} \overset{def}{=} x + D_n$ and define a set of indices

$$I = \{(n, x) : \text{for } n \in \mathbb{Z},\ x \text{ runs through representatives of } K_\infty/T^{-n-1}\mathcal{O}_\infty\},$$

where $\mathcal{O}_\infty$ is the ring of integers in K_∞. Then $\{D_{(n,x)}\}$ is a pure covering of Ω, i.e.

$$\Omega = \bigcup_{(n,x)\in I} D_{(n,x)}.$$

5.2 Separatedness and Properness of Rigid Analytic Spaces

We wish to recognize rigid spaces over C as the analytification of some smooth irreducible projecive variety over C. For a smooth irreducible projective curve X over any field k with a discrete valuation in particular,

$$\{\text{analytic reductions of } X\} \overset{1-1}{\leftrightarrow} \left\{ \begin{array}{c} \text{projective schemes } \mathcal{X}/k^0 : \mathcal{X} \text{ flat over } k^0 \\ \& \text{ generic fiber } \mathcal{X} \times_{k^0} k \cong X \end{array} \right\}.$$

Definition 5.2.1 ([FvdP04, 4.10.1]). *A morphism $f : Z \to X$ of rigid spaces over C is a **closed immersion** if there exists a coherent sheaf of ideals $\mathcal{I}$ on X defining a closed analytic subspace Y of X such that f factors as $Z \xrightarrow{g} Y \to X$ with g an isomorphism. A rigid space X over C is called **separated** if the diagonal morphism $\Delta : X \to X \times_C X$ is a closed immersion.*

We characterize separatedness with the following theory.

Theorem 5.2.2 ([FvdP04, Page 111](Criterion for Separatedness)). *A rigid space X over C is separated if and only if X has an admissible affinoid covering $\{X_i\}$ such that for all $i \neq j$ with $X_i \cap X_j \neq \varnothing$, the intersection $X_i \cap X_j$ is affinoid and the canonical map*

$$\mathcal{O}_X(X_i)\hat{\otimes}_C\mathcal{O}_X(X_j) \to \mathcal{O}_X(X_i \cap X_j)$$

is surjective.

Definition 5.2.3 ([FvdP04, 4.10.2]). *An affinoid subset Y_1 of an affinoid space Y_2 is said to lie in the **interior** of Y_2, denoted $Y_1 \subset\subset Y_2$, if $Y_2 = \mathrm{Sp}(A)$ where A has a presentation for form*

$$A = C\langle z_1, \cdots, z_n\rangle = C\langle Z_1, \cdots, Z_n\rangle/(f_1, \cdots, f_s)$$

and there is some $\rho < 1$ such that $Y_1 \subset \{y \in Y_2 : |z_i(y)| \leqslant \rho \text{ for all } i\}$. Equivalently,

Y_1 is mapped to a single point under the canonical reduction $Y_2 \to \overline{Y_2}^c$.

*A separated rigid space X is **proper** if there exist two finite admissible affinoid covers $\{X_i\}_{i=1,\cdots,n}$ and $\{X_i'\}_{i=1,\cdots,n}$ with $X_i \subset\subset X_i'$ for all i.*

We can think of a relative notion of proper rigid morphisms between rigid spaces: A separated morphism $f : X \to Y$ is **proper** if Y has an admisisble affinoid covering $\{Y_j\}$ such that for each j there exist two affinoid coverings $\{X_i\}_{i=1,\cdots,n}$ and $\{X_i'\}_{i=1,\cdots,n}$ of $f^{-1}Y_j$ such that $X_i \subset\subset_{Y_j} X_i'$ for all i. We mean by this that if $Y_j = \mathrm{Sp}(B)$ and $X_i' = \mathrm{Sp}(A')$, then A' has a presentation

$$A' = B\langle t_1, \cdots, t_\alpha\rangle = B\langle T_1, \cdots, T_\alpha\rangle/(f_1, \cdots, f_b)$$

such that X_i is contained in $\{x \in X_i' : |t_s(x)| \leqslant \rho \text{ for all } s\}$ for some $\rho < 1$.

Some observations from [FvdP04, Pages 111 – 112]:

- A rigid space X is proper if and only if the canonical morphism $X \to \mathrm{Sp}(C)$ is proper.
- If the rigid space X is proper, then all morphisms $f : X \to Y$ are proper.
- Any finite morphism of separated rigid spaces is proper.

A major idea we will use:

Theorem 5.2.4 ([FvdP04, Theorem 4.10.3](Kiehl))**.**

1. *Let X be a proper rigid space over C. Then the cohomology groups $H^i(X, \mathcal{F})$ of any coherent sheaf $\mathcal{F}$ on X are finite dimensional C-vector spaces.*

2. *Let $f : X \to Y$ be a proper morphism of rigid spaces, suppose that Y is separated and let $\mathcal{S}$ be any coherent sheaf on X. Then the direct image $f_*\mathcal{S}$ and all higher direct images $R^i f_*\mathcal{S}$ are coherent sheaves on Y. In particular the image $f(X)$ is a closed analytic subspace of Y.*

Now we come to the point of this discussion.

Theorem 5.2.5 ([FvdP04, Theorem 4.10.6](Recognizing projective varieties))**.** *Suppose X is a rigid space over C such that:*

- *X is reduced, separated and proper over C*
- *There is a C-vector space V of finite dimension $n \geqslant 1$ consisting of meromorphic functions on X such that*

  1. *The coherent subsheaf $\mathcal{L}$ of the sheaf of meromorphic functions $\mathcal{M}$ generated over $\mathcal{O}_X$ by V is a line bundle. This condition means there exists a morphism $\phi : X \to \mathbb{P}^{n,an}_C$ of rigid spaces, where if $f_0, \cdots, f_n$ is a basis for V then ϕ is given by $x \mapsto (f_0(x) : \cdots : f_n(x))$. Note that since C is complete and algebraically closed we do not need to extend ϕ to the base change of X to some completed closure.*

     We ask only that

  2. *ϕ is injective and*
  3. *the tangent map $(d\phi)_x$ is injective for every $x \in X$.*

Then X is the analytification of some projective variety over C.

Example 5.2.6. *We have seen that the action by $\Gamma \leqslant \mathrm{GL}_2(A)$ a congruence subgroup on Ω by Möbius transformations has finite stabilizer for each $z \in \Omega$, and $\Gamma\backslash\Omega$ is a rigid analytic space.*

In particular this is the analytification of the smooth, irreducible, algebraic, affine curve Y_Γ, with smooth projective model X_Γ whose C-points are $\Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$ by [GR96, Theorem 2.2.1].

5.3 Points on a Rigid Analytic Space

Why do we consider the points on a rigid analytic space? It turns out that G-topology is not local enough in the sense that there are nonzero abelian sheaves $\mathcal{F}$ on a rigid space X such that the stalks $\mathcal{F}_x = 0$ for every $x \in X$. Evidently, the set of ordinary points on X is too small. What then should the points be?

We consider the case of an affinoid space and note there are sufficient glueing theorems that one can recover rigid spaces in general from this discussion with enough moxy.

Let $X = \mathrm{Sp}(A)$ be a C-affinoid space. For any abelian sheaf $\mathcal{F}$ on X and any point $x \in X$ we can form the stalk $\mathcal{F}_x$, and for the category of coherent sheaves on X this set is satisfactory in that:

- the functor $\mathcal{F} \mapsto \mathcal{F}_x$ is exact,
- a coherent sheaf $\mathcal{F} = 0$ if the stalks $\mathcal{F}_x = 0$ for all $x \in X$, and

- for any $x \in X$ there exists some coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}_y = 0$ if and only if $y \neq x$.

Definition 5.3.1 ([FvdP04, 7.1.1]). *Consider $X = \mathrm{Sp}(A)$ with the weak G-topology. A **filter** p on X is a collection of admissible subsets such that*

1. *$\varnothing, X \in p$*
2. *If $U_1, U_2 \in p$ then $U_1 \cap U_2 \in p$*
3. *If $U \in p$ and $U \subset V$, then $V \in p$*

   *and we say the filter p is a **prime filter** if*

4. *If $U \in p$ and $\{U_i\}_{i \in I}$ is an admissible covering of U, there is some $i \in I$ such that $U_i \in p$.*

Remark 5.3.2. *The last condition is equivalent to: if U_1, U_2 are admissible and $U_1 \cup U_2 \in p$ then either $U_1 \in p$ or $U_2 \in p$*

The set of filters is ordered by inclusion, so by Zorn's lemma every filter is contained in some **maximal filter**. Maximal filters are prime, and we write

$$\mathcal{P}(X) \overset{def}{:=} \{\text{prime filters}\} \supset \{\text{maximal filters}\} \overset{def}{=:} \mathcal{M}(X).$$

An essential idea for this section is the following:

$$\boxed{\text{Any point } x \in X \text{ induces a maximal filter } \{U \subset X : U \text{ admissible } \& \ x \in U\}.}$$

Let $\mathcal{F}$ be any abelian sheaf and p any prime filter. We can define the stalk $\mathcal{F}_p$ to be the direct limit of the $\mathcal{F}(U)$ with $U \in p$. Then we have the following:

Lemma 5.3.3 ([FvdP04, Page 193])**.**

1. *For admissible $U \subset X$, there is a natural inclusion $\mathcal{P}(U) \subset \mathcal{P}(X)$.*

2. *If $U, V, \{U_i\}_{i\in I}$ are admissible then*

   (a) $\mathcal{P}(U) \subset \mathcal{P}(V) \iff U \subset V$

   (b) If $\cup_{i\in I}\mathcal{P}(U_i) = \mathcal{P}(X)$ then $\cup_{i\in I}U_i = X$ (note the converse is typically false)

Definition 5.3.4. *A topology on $\mathcal{P}(X)$ has a basis $\{\mathcal{P}(U) : U \text{ admissible}\}$ of open sets.*

We consider the categories of abelian sheaves next.,

Theorem 5.3.5 ([FvdP04, Theorem 7.1.2])**.** *Let Ab_X and $\mathrm{Ab}_{\mathcal{P}(X}$ be the abelian categories of abelian sheaves on X and $\mathcal{P}(X)$ respectively.*

1. *$\mathcal{P}(X)$ is a quasi-compact topological space and is not Hausdorff if $\dim X \geqslant 1$,*

2. *For every abelian sheaf $\mathcal{F}$ on $\mathcal{P}(X)$, the presheaf $\sigma_*\mathcal{F}$ defined by $(\sigma_*\mathcal{F})(U) = \mathcal{F}(\mathcal{P}(U))$ for admissible U is a sheaf, and*

   for every prime filter $p \in \mathcal{P}(X)$ the canonical map $(\sigma_*\mathcal{F})_p \to \mathcal{F}_p$ is an isomorphism,

3. *$\sigma_* : \mathrm{Ab}_{\mathcal{P}(X)} \to \mathrm{Ab}_X$ is an equivalence of categories.*

With all of this theory, we can finally come to the correct formulation of points on a rigid analytic space.

Note that the map $X \to \mathcal{P}(X)$ which associates to a point $x \in X$ the maximal filter $\{U : U \text{ admissible } \& \ U \ni x\}$ does not make perfect sense topologically. We consider instead X as a rigid site X_{rigid}. The objects of this site are admissible subsets of X, and

$$\operatorname{Mor}(U, V) = \begin{cases} \varnothing, & U \not\subset V \\ \{U \hookrightarrow V\}, & o/w, \end{cases}$$

i.e. morphisms are at most the single inclusion of U into V. This means the G-topology on X is also a G-topology on X_{rigid}. It is not hard to verify the axioms of a site Definition 3.2.3 for these covers since isomorphisms are inclusions, composition of inclusions are inclusions and fiber-products are well-behaved with respect to injective maps.

Similarly, we can consider the topological space $\mathcal{P}(X)$ as a site $\mathcal{P}(X)_{\text{top}}$. Then we have a morphism of sites $\sigma : \mathcal{P}(X)_{\text{top}} \to X_{\text{rigid}}$ and σ_* is an induced functor between the categories of abelian sheaves on these sites.

With this framework, we have the following consequences of Theorem 5.3.5:

Corollary 5.3.6 ([FvdP04, Page 194]).

- *If p is a prime filter, then the associated functor from abelian sheaves on X to abelian groups given by $\mathcal{F} \mapsto \mathcal{F}_p$ is exact.*
- *An abelian sheaf $\mathcal{F}$ on X is 0 if the stalks $\mathcal{F}_p = 0$ for all prime filters p.*
- *For a given prime filter p there exists an abelian sheaf $\mathcal{F}$ on X such that the stalk $\mathcal{F}_{q'}$ at the prime filter q' is 0 if and only if $q' = p$.*

That is:

The prime filters $\mathcal{P}(X)$ are the "correct" collection of points for the rigid analytic space X.

Example 5.3.7. *On a given Drinfeld modular curve X_Γ, elliptic points, which are no more than points $z \in \Omega$ with stabilizers*

$$\Gamma_z = \left\{ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \frac{az+b}{cz+d} = z, ad - bc \in \mathbb{F}_q^\times \right\}$$

strictly larger than $\mathbb{F}_q^\times \cong \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) : \alpha \in \mathbb{F}_q^\times \right\}$ are ordinary C-points, as these points are well-defined "ordinary" points on the rigid analytic space $Y_\Gamma^{an} = Y_\Gamma(C) = \Gamma\backslash\Omega$. On the log stacky moduli curve $\mathscr{X}_\Gamma(\Delta)$, where Δ is the $\mathbb{Q}$-coefficient Weil divisor of cusps for Γ (the points of $\Gamma\backslash\mathbb{P}^1(K)$ are cusps), elliptic points are stacky points.

The other stacky points are the cusps of a Drinfeld modular curve X_Γ.

algebraically	*analytically*
$X_\Gamma - Y_\Gamma$	*the orbits* $\Gamma\backslash\mathbb{P}^1(K)$

so these are points strictly on the smooth projective model of the affine algebraic curve and correspond to a compactification of $\Gamma\backslash\Omega$. See [Pin21] for more details about the compactification of Drifneld moduli schemes algebraically, and see [BN05] for more algebraic details for stacky curves.

5.4 Rigid GAGA

We begin with a special case.

Theorem 5.4.1 ([FvdP04, Theorem 4.10.5](GAGA for rigid $\mathbb{P}_C^{n,\mathrm{an}}$))**.** *There is a functor $\mathcal{F} \mapsto \mathcal{F}^{an}$ from the category of coherent sheaves on $\mathbb{P}_C^n$ to the category of coherent sheaves on the rigid space $\mathbb{P}_C^{n,an}$. This is an equivalence of abelian categories and commutes with the formation of cohomology groups. In particular every closed analytic subspace of $\mathbb{P}_C^{n,an}$ is the analytification of some closed subspace of $\mathbb{P}_C^n$.*

Example 5.4.2. *The rigid analytic quotient spaces $\Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$ are known to be one-dimensional and finite covers of $\mathrm{GL}_2(A)\backslash(\Omega \cup \mathbb{P}^1(K)) \cong \mathbb{P}_C^{1,an}$ by [Gek01, Page 170]. As such, there is some embedding of these spaces by a line bundle (some multiple of the canonical bundle according to the classification of the curve, e.g. Petri's theorem) and so the embedded curve is a closed analytic subspace of an analytic projective space. Then such quotient spaces are the analytification of embedded projective Drinfeld modular curves X_Γ.*

With this rather crude argument we at least motivate the idea that Drinfeld modular curves (the stacky curves $\mathscr{X}_\Gamma$ which we consider in Chapter 6) have a coarse space X_Γ whose analytification is a familiar rigid analytic space $\Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$. We consider further both the mechanics of rigid GAGA itself, and the theory of covers of $\mathbb{P}^{1,\mathrm{an}}$.

Let M be a coherent sheaf on the algebraic curve $\mathbb{P}_C^1$ over C. Then there is an associated coherent sheaf M^{an} on $\mathbb{P} \overset{def}{=} \mathbb{P}_C^{1,\mathrm{an}}$ given as follows:

1. for an affinoid X, choose a Zariski open S with $X \subset S \neq \mathbb{P}^1_C$

2. $M^{\mathrm{an}}(X) \overset{def}{=} \mathcal{O}(X) \otimes_{\mathcal{O}_{\mathrm{alg}}(S)} M(S)$, where $\mathcal{O}_{\mathrm{alg}}(S) \subset C(z)$ denotes the ring of regular functions on S. Further, $M^{\mathrm{an}}(\mathbb{P}) \overset{def}{=} M(\mathbb{P}^1_C)$.

A morphism $f : M_1 \to M_2$ of coherent sheaves on $\mathbb{P}^1_C$ induces a morphism f^{an} between coherent sheaves M_1^{an} and M_2^{an} on $\mathbb{P}$.

At a point $a \in \mathbb{P}^1_C$, consider the local parameter $t = z - a$ or $t = z^{-1}$. There are three important local rings:

1. the algebraic local ring $C[t]_{(t)}$

2. the analytic local ring $C\{t\}$ consisting of convergent power series, &

3. $C[[t]]$ the ring of formal power series

and

$$C[t]_{(t)} \subset C\{t\} \subset C[[t]],$$

where the final local ring is the completion of the first two. Therefore the inclusion $C[t]_{(t)} \subset C\{t\}$ is faithfully flat.

5.5 Rigid Stacky GAGA

Now we consider promoting the basic rigid GAGA from Theorem 5.4.1 to stacks.

5.5.1 Context

In this section we explain a version of GAGA for Deligne-Mumford rigid analytic, and algebraic Deligne-Mumford stacks. We begin by collecting some intermediate theory from the literature for context.

First, we recall that DM stacks are **Artin stacks**. See [Art74] for the first discussion of what we now call Artin stacks and the papers [AOV07] and [AOV10] for a more complete discussion of such stacks. Second, we recall that

$$\boxed{\text{rigid analytic spaces are adic spaces or Berkovich spaces}}$$

where adic and Berkovich spaces are each generalizations of rigid analytic spaces. We remark that because of the enhanced levels of generality for these sites over non-archimedean fields it is a fair question to ask, "why do we still work on rigid analytic spaces?" Indeed, though sometimes this more classical theory gives us more intuitive notions, say of points e.g. we still need to deal with G-topologies for rigid spaces, so pedagogically a rigid analytic space is about as hard to think about as the more general adic and Berkovich spaces. Our reason is this: we do not want to reinvent the wheel. The theory of stacks for these sites is sufficiently well developed for us to use it for our program of computing algebras of Drinfeld modular forms via geometric invariants of Drinfeld modular curves as in [VZB22]. That is, our principle is conservation of pain: we aim to state as little as possible to do as much as is necessary.

5.5.2 Rigid Stacks

We need a precise notion of a rigid analytic stack for rigid stacky GAGA. Let us compare Definition 6.3.4 to some formulations in the literature to begin. One of the first times the phrase "rigid analytic stack" appears is [Iwa06]. After a conjecture on [Iwa06, Page 21], there is the following remark: "roughly speaking, this conjecture says that in rigid geometry, the existence of local deformation theory implies the global moduli stack which is represented by a rigid analytic stack." Then in 2017 the Ph.D. thesis [War17] develops a theory of Artin stacks on adic spaces. However, we follow the more recent [EGH23, Section 5.1.7] for expedience.

Fix L some finite extension of $\mathbb{Q}_p$. Let Rig_L denote the category of rigid analytic spaces over L. Equp Rig_L with the **Tate-fpqc** topology (see [CT07, 2.1]). The covers in this topology are generated by the admissible Tate coverings and the morphisms $\mathrm{Sp}(A) \to \mathrm{Sp}(B)$ for faithfully flat morphisms of affinoid algebras $B \to A$. By [Con06, Theorem 4.2.8] all representable functors in this topology are sheaves and coherent sheaves satisfy descent.

A **stack on** Rig_L is a category fibered in groupoids which satisfies descent for the Tate-fpqc topology. We use one last intermediate definition before we can rigorously define the kind of rigid analytic stacks which are our focus.

Definition 5.5.1 ([EGH23, 5.1.8]). *A **quasi-analytic space** is a sheaf $\mathcal{F}$ on Rig_L such that the diagonal $\Delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \times_L \mathcal{F}$ is representable and there exists an étale surjection $U \to \mathcal{F}$ for a representable sheaf U.*

As in [EGH23, Remark 5.1.9.(i)] we do not need this level of generality since by [CT07, Theorem 1.2.2] every quasi-analytic space is representable by a rigid analytic space.

Definition 5.5.2 ([EGH23, 5.1.10]). *A **rigid analytic Artin stack** is a stack $\mathscr{X}$ on Rig_L such that the diagonal $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_L \mathscr{X}$ is representable by a quasi-analytic space, and there exists some rigid analytic space U and a smooth surjective map $U \to \mathscr{X}$.*

We define rigid analytic Deligne-Mumford stacks now, but return to this matter in Definition 6.3.4.

Definition 5.5.3. *A **rigid analytic Deligne-Mumford stack** is a rigid analytic Artin stack $\mathscr{X}$ such that the diagonal $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_L \mathscr{X}$ is representable by a rigid analytic space, quasi-compact and separated for the Tate-fpqc topology.*

5.5.3 GAGA Theorem

Now we can make sense of a well-defined rigid and stacky GAGA following the theory of [PY16]. The next two results are analogs of [Ser56, Theorems 2 and 3]. First, we describe analytification with the following Lemma.

Lemma 5.5.4 ([PY16, Lemma 7.2]). *Let A be a k-affinoid algebra, for k some non-achimedean field. Let $\mathscr{X}$ be an algebraic stack locally of finite presentation over* $\mathrm{Spec}(A)$*. Suppose that for $\mathcal{F} \in \mathcal{O}_{\mathscr{X}} - \mathrm{Mod}$ we have $\mathcal{F} \cong \lim_{\tau \geqslant -n} \mathcal{F}$. Then the analytification functor $(-)^{an}$ commutes with this limit.*

We have an equivalence of categories between algebraic and rigid analytic coherent sheaves which we describe with the following two results.

Proposition 5.5.5 ([PY16, Proposition 7.3]). *Let A be a k-affinoid algebra, for k some non-achimedean field. Let $\mathscr{X}$ be an algebraic stack locally of finite presentation over* $\mathrm{Spec}(A)$. *Suppose that $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(\mathscr{X})$ are coherent sheaves on $\mathscr{X}$. Then there is an equivalence of categories given by the natural map*

$$\mathrm{Map}_{\mathrm{Coh}(\mathscr{X})}(\mathcal{F}, \mathcal{G}) \xrightarrow{\cong} \mathrm{Map}_{\mathrm{Coh}(\mathscr{X}^{an})}(\mathcal{F}^{an}, \mathcal{G}^{an}).$$

Finally, the main result we need is the following.

Theorem 5.5.6 (Theorems [PY16, 7.4 and 7.5]). *Let A be a k-affinoid algebra, for k some non-achimedean field. Let $\mathscr{X}$ be a proper algebraic stack over* $\mathrm{Spec}(A)$. *The analytification functor on coherent sheaves induces an equivalence of categories*

$$\mathrm{Coh}(\mathscr{X}) \xrightarrow{\cong} \mathrm{Coh}(\mathscr{X}^{an}).$$

Remark 5.5.7. *In particular the Theorems of Porta-Yu apply to both 1-categories and ∞-categories. We can use the slightly weaker statement given above to avoid defining ∞-categories, which do not show up for us.*

Chapter 6

Drinfeld Modular Curves and Forms

This chapter describes moduli spaces of Drinfeld modules and Drinfeld modular forms. In the next chapter we state and prove our main results about the connections between the geometry of these two kinds of objects, and so our task now is to provide all of the tools we will need in our coming arguments. We have finally established all of the terminology we will need in preceeding chapters, so that with the puzzle pieces we describe in this chapter, in the next we can fit together a complete description.

6.1 Drinfeld Modular Forms

In this section we introduce Drinfeld modular forms. The technical conditions of the rigid analytic space in which we work makes it necessary to introduce some facts about the projective line $\mathbb{P}^1(C)$ before we begin in earnest on a study of modular forms. We discuss rigid analytic spaces in more detail in the following sections.

Definition 6.1.1. *Let $\overline{\pi} \in K_\infty(\sqrt[q-1]{-T})$ be a fixed choice of the Carlitz period (recall*

Example 4.3.6). Then we define a ***parameter at infinity***

$$u(z) \overset{def}{=} \frac{1}{e_{\bar{\pi}A}(\bar{\pi}z)} = \frac{1}{\bar{\pi}e_A(z)} = \bar{\pi}^{-1}\sum_{a\in A}\frac{1}{z+a}.$$

Remark 6.1.2. *In the Drinfeld setting, $\overline{\pi}$ plays the role of the constant $2\pi i \in \mathbb{C}$ in the parameter $q = e^{2\pi iz}$ at infinity from the classical setting. That is, it is a normalization factor so that the series expansion coefficients for modular forms at cusps are elements of A.*

One fact about this parameter which we will use later in our consideration of modular forms is the following.

Lemma 6.1.3 ([Gek99, Page 10])**.** *Let $\alpha \in \mathbb{F}_q^\times$. Then $u(\alpha z) = \alpha^{-1}u(z)$.*

Proof. For any $\alpha \in \mathbb{F}_q^\times$, the lattices $\Lambda = A\omega_1 + A\omega_2 \subset C$ and $\alpha \cdot \Lambda$ are similar. Furthermore, Λ is similar to $(\omega_1/\omega_2)A + A$, where $z = \omega_1/\omega_2 \in \Omega$. So the exponential functions $e_{\bar{\pi}\alpha A}(\bar{\pi}\alpha z)$ and $e_A(z)$ for $z \in \Omega$ differ by a factor of $\bar{\pi}\alpha^{-1}$. □

We mention some terminology which is part of the definition of a Drinfeld modular form.

Definition 6.1.4. *We say a function $f : \Omega \to C$ such that $f(\gamma z) = \det(\gamma)^{-l}(cz + d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & b \end{smallmatrix}\right) \in \Gamma$, where $k \in \mathbb{Z}_{\geqslant 0}$, $l \in \mathbb{Z}/(q-1)\mathbb{Z}$ and $\Gamma \leqslant \mathrm{GL}_2(A)$ is a congruence subgroup, is* ***weakly modular of weight*** *k* ***and type*** *l* ***for*** *Γ.*

Now we are able to define a fundamental object of study for this thesis.

Definition 6.1.5 ([Gek86, Definition (3.1)])**.** *Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup. A* ***modular form*** *of* ***weight*** *$k \in \mathbb{Z}_{\geqslant 0}$ and* ***type*** *$l \in \mathbb{Z}/((q-1)\mathbb{Z})$ is a holomorphic function $f : \Omega \to C$ such that*

1. $f(\gamma z) = \det(\gamma)^{-l}(cz+d)^k f(z)$ *for all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & b \end{smallmatrix}\right) \in \Gamma$, *and*

2. f *is holomorphic at the cusps of* Γ, *i.e. at the representatives for the orbits of* $\mathbb{P}^1(K)$ *under the action of* Γ.

Remark 6.1.6. *We have gotten somewhat ahead of ourselves by describing cusps of* Γ, *a topic which we return to in more detail in Definition 6.2.5. For now we make the simplifying assumption that, like* $\mathrm{GL}_2(A)$ *itself,* Γ *acts transitively on* $\mathbb{P}^1(K)$. *We pick* ∞ *to be a representative for this orbit. That is, for now we say the unique cusp of* Γ *is* ∞, *which is enough to make some comments on the role of cusps in the definition of a Drinfeld modular form.*

There are several interpretations of the second condition about holomorphy at ∞, *two of which are particularly helpful for intuition and for the proof the main theorem respectively:*

1. *[Gek01,* (2.2.iii)*] The condition is equivalent to* f *being bounded on* $\{z \in \Omega : |z|_\infty \geqslant 1\}$, *where* $|\cdot|_\infty$ *is the* ∞*-adic absolute value, in any case when* Γ *has a single cusp;*

2. *[Gek99, Definition* 3.5.(iii)*]* f *has a series expansion at cusps:*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n u(z)^n, \ \ a_n \in C,$$

where u *is the parameter at* ∞, *with a positive radius of convergence. The second condition means that* $a_n = 0$ *for all* $n < 0$.

Remark 6.1.7. *The observation from [Gek88, Definition* (5.7)*] that if* f *is Drinfeld* *modular form, then $f(z + b) = f(z)$ for $b \in A$ means that although not literally a Fourier series, the series expansion of a modular form at the cusps of some congruence subgroup is "morally" the Drinfeld setting equivalent to a Fourier series.*

We introduce some terminology and notation respectively in the next definition.

Definition 6.1.8. *Write $M_{k,l}(\Gamma)$ for the finite-dimensional C-vector space of Drinfeld modular forms for $\Gamma \leqslant \mathrm{GL}_2(A)$ with weight k and type l. The* ***algebra*** *$M(\Gamma)$* ***of modular forms for*** *Γ is*

$$M(\Gamma) = \bigoplus_{\substack{k \geqslant 0 \\ l \pmod{q-1}}} M_{k,l}(\Gamma)$$

since $M_{k,l} \cdot M_{k',l'} \subset M_{k+k',l+l'}$.

Now we can introduce some non-trivial facts about Drinfeld modular forms.

Lemma 6.1.9 ([Gek88, Remark 5.8.iii])**.** *Suppose $f(z) \in M_{k,l}(\Gamma)$ has a u-series expansion $f(z) = \sum_{n \geqslant 0} a_n u^n$. Then the coefficients a_i uniquely determine f.*

Proof. Although the u-series may not converge on all of Ω and only for $|u(z)|$ small, since Ω is connected in the rigid analytic sense, the result follows from having a unique u-series anywhere within Ω. $\square$

The weight and type of Drinfeld modular forms are not independent quantities in the sense of the following fact.

Lemma 6.1.10 ([Gek88, Remark (5.8.i)])**.** *If $M_{k,l}(\Gamma) \neq 0$, then $k \equiv 2l \pmod{q-1}$.*

Proof. Let $\gamma = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ for some $\alpha \in \mathbb{F}_q^\times$. By assumption Γ contains the matrices of form $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha' \end{smallmatrix}\right)$ for all $\alpha, \alpha' \in \mathbb{F}_q^\times$, therefore $\gamma \in \Gamma$. If f is a non-zero modular form for Γ

of weight k and type l then

$$f(\gamma z) = f\left(\frac{\alpha z}{\alpha}\right) = f(z) = \alpha^k \alpha^{-2l} f(z),$$

so $\alpha^k = \alpha^{2l}$ in $\mathbb{F}_q^\times$ and we conclude $k \equiv 2l \pmod{q-1}$. $\square$

Example 6.1.11. *Some famous Drinfeld modular forms are the* $\mathrm{GL}_2(A)$*-forms:* g *of weight* $q-1$ *and type* 0, Δ *of weight* q^2-1 *and type* 0, *and* h *of weight* $q+1$ *and type* 1. *We know from Goss and Gekeler respectively, see for example [Gek99, Theorem* (3.12)*], that*

$$\bigoplus_{k\geqslant 0} M_{k,0}(\mathrm{GL}_2(A)) = C[g,\Delta] \quad \textit{and} \quad \bigoplus_{\substack{k\geqslant 0\\ l \pmod{q-1}}} M_{k,l}(\mathrm{GL}_2(A)) = C[g,h].$$

Example 6.1.12 ([Gek88])**.** *The function*

$$E(z) \overset{def}{=} \overline{\pi}^{-1} \sum_{\substack{a\in A\\ monic}} \left(\sum_{b\in A} \frac{a}{az+b} \right)$$

is an analog to an Eisenstein series of weight 2 *over* $\mathbb{Q}$, *and we can define a Drinfeld modular form*

$$E_T(z) \overset{def}{=} E(z) - TE(Tz)$$

of weight 2 *and type* 1 *for* $\Gamma_0(T)$, *the congruence subgroup of* $\mathrm{GL}_2(A)$ *containing matrices* $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ *with* $c \equiv 0 \pmod{T}$.

As in [Gek99, Definition (3.5)], for f some Drinfeld modular form, we let $v_z(f)$ denote the vanishing order of f at $z \in \Omega$ and $v_\infty(f)$ denote the vanishing order of f

at ∞. Then from [Gek99, Equation (3.10)]:

$$\sum_{z \in GL_2(A)\backslash\Omega}^{*} v_z(f) + \frac{v_e(f)}{q+1} + \frac{v_\infty(f)}{q-1} = \frac{k}{q^2-1},$$

where $\sum^*$ denotes a sum over non-elliptic classes of $\mathrm{GL}_2(A)\backslash\Omega$. By a non-elliptic class we mean some point in the quotient whose stabilizer under $\mathrm{GL}_2(A)$ is strictly larger than $\mathbb{F}_q^\times$, which we discuss further in Definition 6.2.7.

6.2 Drinfeld Modular Curves

Let us consider some moduli spaces of rank 2 Drinfeld modules (possibly with some extra torsion information), first as rigid analytic spaces, then as moduli schemes, and finally as log stacky curves.

For the well-definedness of Drinfeld modular curves, we consider some analytic properties of Ω. Since $\Omega = \mathbb{P}^1(C) - \mathbb{P}^1(K_\infty)$, and $\mathbb{P}^1(K_\infty)$ is compact in the rigid analytic topology, we know from [GR96, Section 1.2] that Ω is a rigid analytic space. The action by $\Gamma \leqslant \mathrm{GL}_2(A)$ a congruence subgroup on Ω by Möbius transformations has finite stabilizer for each $z \in \Omega$, and as in [GR96, Sections (2.5) and (2.6)], $\Gamma\backslash\Omega$ is a rigid analytic space.

Recall that for any scheme S of locally finite type over a complete, non-archimedian field of finite characteristic p, there is a rigid analytic space S^{an} whose points coincide with those of S as sets. In fact, there is an **analytification functor** from the category of schemes over C to the category of rigid analytic spaces, so if X is a smooth algebraic curve over C, then there is a rigid analytic space X^{an} whose points are in bijection with the C-points of X.

For example, we have the following description.

Theorem 6.2.1 ([Dri74]). *Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup. There exists a smooth, irreducible, affine algebraic curve Y_Γ over C such that $\Gamma\backslash\Omega$ and the underlying (rigid) analytic space Y_Γ^{an} of Y_Γ are canonically isomorphic as rigid analytic spaces over C.*

Remark 6.2.2. *This underlying rigid analytic space is the analytification of Y_Γ as in [FvdP04, Example 4.3.3].*

Definition 6.2.3. *We call the affine curves Y_Γ with analytification $Y_\Gamma^{an} \cong \Gamma\backslash\Omega$ as rigid analytic spaces over C **affine Drinfeld modular curves.** Since Y_Γ is smooth and affine, it admits a smooth projective model which X_Γ which is a **projective Drinfeld modular curve**.*

Remark 6.2.4. *In the spirit of [VZB22, Section 6.2], we say a projective Drinfeld modular curve X_Γ is the **algebraization** of some rigid analytic space $\Gamma\backslash(\Omega\cup\mathbb{P}^1(K)) = X_\Gamma^{an}$, whose points are in bijection with the C-points of the projective Drinfeld modular curve X_Γ.*

Let $X_\Gamma^{\mathrm{an}} \overset{def}{=} \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$ denote a rigid analytic, projective Drinfeld modular curve for some congruence subgroup $\Gamma \leqslant \mathrm{GL}_2(A)$. Let $X_\Gamma = (X_\Gamma^{\mathrm{an}})^{\mathrm{alg}}$ denote the corresponding algebraic Drinfeld modular curve whose C-points are in bijection with X_Γ^{an}. This modular curve is not a stacky curve since there is a uniform μ_{q-1} stabilizer which we know from the moduli interpretation - each point is fixed by

$Z(GL_2(A)) = \{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) : \alpha \in \mathbb{F}_q^\times \} \cong \mathbb{F}_q^\times$. However, as a scheme, X_Γ is the coarse space of a stacky curve $\mathscr{X}_\Gamma$ given by the stack quotient $[X_\Gamma / Z(\mathrm{GL}_2(A))]$. Furthermore, if $\mathcal{M}^2_\Gamma$ denotes (Laumon's) Deligne-Mumford stack representing the corresponding moduli problem, then every point of $\mathcal{M}^2_\Gamma$ has a stabilizer containing (at least) $\mathbb{F}_q^\times$. Then $\mathcal{M}^2_\Gamma$ is a μ_{q-1}-gerbe over $\mathscr{X}_\Gamma$, i.e. $\mathscr{X}_\Gamma = \mathcal{M}^2_\Gamma /\!/ \mu_{q-1}$ is a rigidification of $\mathcal{M}^2_\Gamma$:

$$\mathcal{M}^2_\Gamma \to \mathscr{X}_\Gamma \to X_\Gamma.$$

When we discuss **stacky Drinfeld modular curves** we mean a curve $\mathscr{X}_\Gamma$ as in this paragraph, that is the rigidification of some moduli problem (i.e. of one of Laumon's gerbes).

Next we consider some special points on Drinfeld modular curves.

Definition 6.2.5. *Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup, let $Y_\Gamma^{an} = \Gamma \backslash \Omega$ and let $X_\Gamma^{an} = \Gamma \backslash (\Omega \cup \mathbb{P}^1(K))$. A **cusp of X_Γ^{an}** is a point of $X_\Gamma^{an} - Y_\Gamma^{an}$.*

Remark 6.2.6. *As sets, $X_\Gamma^{an} = \Gamma \backslash (\Omega \cup \mathbb{P}^1(K))$, so since $\mathrm{GL}_2(A)$ acts transitively on $\mathbb{P}^1(K)$ we have*

$$\mathcal{C}_\Gamma \overset{def}{=} \{\textit{cusps of } X_\Gamma^{an}\} \overset{def}{=} \Gamma \backslash \mathbb{P}^1(K) = \Gamma \backslash \mathrm{GL}_2(A)/\mathrm{GL}_2(A)_\infty,$$

where $\mathrm{GL}_2(A)_\infty = \{\gamma \in \mathrm{GL}_2(A) : \gamma(\infty) = \infty\} = \{ \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \}$. On a Deligne-Mumford stacky curve, the stabilizer of each point is a finite cyclic group, so evidently there are too many stabilizing matrices here. We will discuss this in more detail in Section 6.4.3.

Definition 6.2.7.

1. *If $e \in \Omega$ has $(\mathrm{GL}_2(A))_e = \{\gamma \in \mathrm{GL}_2(A) : \gamma(e) = e\}$ strictly larger than $\mathbb{F}_q^\times \cong$*

$\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)\}$ *then* e *is an* ***elliptic point on*** Ω*. In this case,* $\mathrm{GL}_2(A)_e \cong \mathbb{F}_{q^2}^{\times}$.

2. *Let* $\Gamma \leqslant \mathrm{GL}_2(A)$ *be a congruence subgroup. A point* $e \in \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$ *is an* ***elliptic point for*** Γ *if the stabilizer* Γ_e *is strictly larger than the center of* $\mathrm{GL}_2(\mathbb{F}_q)$*:* $\mathbb{F}_q^{\times} \cong \left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) : \alpha \in \mathbb{F}_q^{\times}\right\}$.

Remark 6.2.8. *An elliptic point* e *on* Ω *is a point which is* $\mathrm{GL}_2(A)$*-conjugate to some element of* $\mathbb{F}_{q^2}\backslash\mathbb{F}_q \hookrightarrow \Omega$*. Fix once and for all an elliptic point* $e \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$ *on* Ω*. We write*

$$\mathrm{Ell}(\Gamma) \overset{def}{=} \{\textit{elliptic points of } X_\Gamma^{an}\}.$$

Remark 6.2.9. *Note that Drinfeld modular curves* $\mathscr{X}_\Gamma$*, for* $\Gamma \leqslant \mathrm{GL}_2(A)$ *any congruence subgroup, are* ***tame*** *over* C *in the sense of [VZB22, Example* 5.2.7*]. We may describe* $\mathscr{X}_\Gamma$ *by the stack quotient* $[X_\Gamma/Z(\mathrm{GL}_2(A))]$*, and since* $\gcd(\mathrm{char}(C), \#Z(\mathrm{GL}_2(A))) = 1$ *the quotient is tame.*

Example 6.2.10 (The j-line)**.** *Let* $X(1) = \mathrm{GL}_2(A)\backslash(\Omega \cup \mathbb{P}^1(K))$ *be the "usual"* j*-line. Let* $\mathcal{M}_A^2$ *be (Laumon's) Deligne-Mumford stack representing the moduli corresponding problem. Then* $\mathcal{M}_A^2$ *is a* μ_{q-1} *gerbe over* $\mathscr{X}(1) = [X(1)/Z(\mathrm{GL}_2(A))]$*. In other words,* $\mathscr{X}(1)$ *is a rigidification* $\mathcal{M}_A^2/\!/\mu_{q-1}$*:*

$$\begin{array}{ccccc} \mathcal{M}_A^2 & \xrightarrow{\pi} & \mathscr{X}(1) & \to X(1) \\ \mathbb{P}^1((q-1)^2, q^2-1) & \xrightarrow{\pi} & \mathbb{P}^1(q-1, q+1) & \to \mathbb{P}^1(C). \end{array}$$

6.3 Compactification of Drinfeld Moduli Schemes and Uniformization of Drinfeld Moduli Stacks

6.3.1 Schemes

We start with facts, mostly for the notation, about Drinfeld modules. Let $A = \mathbb{F}_q[T]$ and let $K = \mathrm{Frac}(A)$. For any scheme S over K and E a line bundle on S, write $\mathrm{End}_{\mathbb{F}_q}(E)$ for the ring of $\mathbb{F}_q$-linear endomorphisms of the commutative group scheme underlying E. Any trivialization

$$\mathbb{G}_{a,U} \xrightarrow{\sim} E|_U$$

over an affine subscheme $U = \mathrm{Spec}(R) \subset S$ induces an isomorphism

$$R\{X^q\} \xrightarrow{\sim} \mathrm{End}_{\mathbb{F}_q}(E|_U)$$

as in Definition 4.3.4. Let (E, φ) be a Drinfeld A-module and let N be an ideal of A. Then

$$\varphi[N] \overset{def}{=} \bigcap_{a \in N} \ker(\varphi_a)$$

is an A-module subscheme of E which is finite étale over S. Its sections over any geometric point of S form a free A/N-module of rank r. Further, the A-module

$$V_N^r \stackrel{def}{=} (N^{-1}/A)^{\oplus r},$$

where $N^{-1} \subset K$ is the inverse fractional ideal of N, is also a free A/N-module of rank r. A **level N-structure on** φ is an A-module homomorphism

$$\lambda : V_N^r \to \varphi[N](S)$$

which induces isomorphisms in every fiber.

Now we can describe the moduli spaces of Drinfeld modules of rank r.

Theorem 6.3.1 ([Pin21, 2.1.1]). *There exists a scheme $M_{A,N}^r$ over K and a triple $(E^{univ}, \varphi^{univ}, \lambda^{univ})$ such that*

1. *for any K-scheme S and (E, φ, λ) there exists a unique morphism $f : S \to M_{A,N}^r$ such that $(E, \varphi, \lambda) \cong f^*(E^{univ}, \varphi^{univ}, \lambda^{univ})$, and*

2. *$M_{A,N}^r$ is an irreducible, smooth affine algebraic variety of finite type and dimension $r-1$ over K.*

Example 6.3.2. *$M_{A,N}^2 \cong Y_\Gamma$ the (coarse space of the) Drinfeld modular curve whose C-points are the rigid analytic space $\Gamma\backslash\Omega$, where $\Gamma \leqslant \mathrm{GL}_2(A)$ is a congruence subgroup with conductor N.*

Note that for any two ideals $N \subset N' \subset A$ there is a natural inclusion $V_{N'}^r \subset V_N^r$, so any level N-structure restricts to a level N'-structure. We may apply this when

$N' = (a)$ for any nonzero $a \in A$: let $V_a^r \overset{def}{=} (a^{-1}A/A)^{\oplus r}$. Let R be any K-algebra and let R_{A,V_N^r} denote the quotient ring $R_{V_N^r} \otimes_{\mathbb{F}_q} K$ modulo the ideal

$$\left(\left\{\frac{1}{av} \otimes a - \sum_{v' \in V_a^r} \frac{1}{v - v'} \otimes 1 : a \in \mathrm{Div}(N) \text{ and } v \in V_N^r - V_a^r\right\}\right).$$

The inherent grading of $R_{V_N^r}$ as a K-algebra induces a grading on $R_{V_N^r} \otimes_{\mathbb{F}_q} K$ and each generator for this ideal is homogeneous of degree 1. For any $f \in R_{V_N^r} \otimes_{\mathbb{F}_q} K$ let $[f]$ denote its image in R_{A,V_N^r} and let RS_{A,V_N^r} be the localization of R_{A,V_N^r} obtained by inverting the elements $[\frac{1}{v} \otimes 1]$ for all $v \in \overset{\circ}{V}{}_N^r \overset{def}{=} V_N^r - \{0\}$.

Now that we have the notation we need, we can deal with the problem of compactifying the Drinfeld moduli schemes $Y_\Gamma = M_{A,N}^2$, so we restrict our notation to this case from now on.

Consider $Q_{A,V_N^2} \overset{def}{=} \mathrm{Proj}(R_{A,V_N^2})$, a projective scheme over K with a natural very ample line bundle $\mathcal{O}(1)$ and natural ring homomorphisms $R_{A,V_N^2,n} \to \mathcal{O}(n)(Q_{A,V_N^2})$ for each $n \in \mathbb{Z}$. Together, these mean we have a closed embedding

$$Q_{A,V_N^2} \hookrightarrow Q_{V_N^2} \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}(K).$$

Let

$$\Omega_{A,V_N^2} \overset{def}{=} \mathrm{Proj}(RS_{A,V_N^2}) \cong \mathrm{Spec}(RS_{A,V_N^2,0}).$$

Then since RS_{A,V_N^2} is the localization of R_{A,V_N^2} obtained by inverting a non-empty, finite set of elements of degree 1 we see that Ω_{A,V_N^2} is an affine open subscheme of

Q_{A,V_N^2}. In fact, by [Pin21, Theorem 2.7.6], Ω_{A,V_N^2} is a dense open subscheme of Q_{A,V_N^2}. What is more, we have encountered this subscheme before, in another guise. By [Pin21, 2.6.5], we know $M^2_{A,N} \cong \Omega^2_{A,N}$ is the affine Drinfeld modular curve Y_Γ. Finally, from [Pin21, Section 2.9], we know the scheme $Q^{\text{norm}}_{A,V_N^2}$ is the **Satake compactification** $\overline{M^2_{A,N}}$ of $M^2_{A,N}$, where we denote by $R^{\text{norm}}_{A,V_N^2}$ the integral closure of R_{A,V_N^2} in RS_{A,V_N^2} and

$$Q^{\text{norm}}_{A,V_N^2} \stackrel{def}{=} \text{Proj}(R^{\text{norm}}_{A,V_N^2}).$$

We have seen this projective curve as well, it is the projective model X_Γ from Definition 6.2.3.

6.3.2 Stacks

Let us compare this with the theory of uniformization of Deligne-Mumford curves.

For this discussion, all stacks are assumed to be smooth and separated. We say an **orbifold** is a smooth, DM, (topological, algebraic or rigid analytic) stack which is generically a (topological, algebraic or rigid analytic) space. On any stack, we say an **orbifold point** is some point at which the intertia group jumps. Denote the coarse space of a (topological, algebraic, or rigid analytic) stack $\mathscr{X}$ by X. To avoid 2-isomorphisms, all such morphisms are declared equalities. We will consider the property of stacks which we now introduce.

Definition 6.3.3 ([BN05, Page 4]). *A topological stack $\mathscr{X}$ is* ***uniformizable*** *if its universal cover is a (genuine) topological space.*

Let **Top** be the category of topological spaces with the usual Groethendieck topology where covers are simply topological open covers. A morphism $f : \mathscr{Y} \to \mathscr{X}$ of stacks over **Top** is **representable** if for any map $X \to \mathscr{X}$ with X a topological space, $Y = X \times_{\mathscr{X}} \mathscr{Y}$ is isomorphic to some topological space. A **pre-DM topological stack** is a stack for which there exists some epimorphism $p : X \to \mathscr{X}$ from a topological space such that p is representable by a local homeomorphism. A pre-DM topological stack is a **DM topological stack** if $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable by a closed map with discrete finite fibers. For any topological stack $\mathscr{X}$ there is a genuine topological space, X, the coarse space of $\mathscr{X}$.

Let **Rigid** be the category of rigid analytic spaces, endowed with some G-topology which is at least as fine as the very weak G-topology (recall Example 5.1.4). The 2-category of stacks over **Rigid** contains the category of rigid analytic spaces as a full subcategory. We say a morphism of stacks $f : \mathscr{Y} \to \mathscr{X}$ over **Rigid** is **representable by local homeomorphisms** if for any map $X \to \mathscr{X}$ from a rigid analytic space X to $\mathscr{X}$, the fiber-product $Y = X \times_{\mathscr{X}} \mathscr{Y}$ is isomorphic to a rigid analytic space and the induced map $Y \to X$ is a local homeomorphism of rigid analytic spaces. A stack $\mathscr{X}$ over **Rigid** is a **pre-DM rigid analytic stack** if there exists an epimorphism $p : X \to \mathscr{X}$ with X a rigid analytic space, such that p is representable by local homeomorphisms. A morphism of pre-DM rigid analytic stacks is **representable** if for any map $X \to \mathscr{X}$ from a rigid analytic space that is representable by local homeomorphisms, the fiber product $Y = X \times_{\mathscr{X}} \mathscr{Y}$ is isomorphic to a rigid analytic space.

Definition 6.3.4. *A pre-DM rigid analytic stack $\mathscr{X}$ over a non-achimedean field k is a* ***DM rigid analytic stack*** *if the diagonal $\mathscr{X} \to \mathscr{X} \times_k \mathscr{X}$ is representable by* *closed map with finite fibers.*

Remark 6.3.5. *We earlier defined such a stack in Definition 5.5.3. The condition here that the diagonal is a closed map with finite fibers is equivalent to the separatedness and quasi-compactness from Definition 5.5.3.*

Example 6.3.6. *The Drinfeld moduli spaces, $\mathcal{M}_N^2$ in the notation of [Lau96, Section* (1.4)*] are known to be Deligne-Mumford algebraic stacks of finite type over $\mathbb{F}_p$ by [Lau96, Corollary* 1.4.3*].*

Remark 6.3.7. *Inspired by [BN05, Proposition* 3.5*], we can state a "conjecture" of similar content and form:*

Let $\mathscr{X}$ be a DM rigid analytic stack. Then there is a covering $\{\mathcal{U}_i\}$ of $\mathscr{X}$ by open substacks such that each $\mathcal{U}_i$ is a quotient stack $[X/G]$, where X is a rigid analytic space and G a finite group acting rigid-analytically on X.

This statement is only meant to indicate the question: "what does it mean and take for a statement of this form to be proven?"

Note that there is a well-defined coarse space X for a rigid analytic DM stack $\mathscr{X}$, where by well-defined we mean that X is a rigid analytic space of dimension 1.

Where are the groupoids?

Let $\mathscr{X}$ be a smooth, DM algebraic stack of finite type over C, with $X_1 \rightrightarrows X_0 \times X_0$ an étale groupoid representing it. We define $\mathscr{X}^{\mathrm{an}}$ to be the quotient of the groupoid $X_1^{\mathrm{an}} \rightrightarrows X_0^{\mathrm{an}} \times X_0^{\mathrm{an}}$. The same thing allows us to get a topological stack from a rigid analytic stack. The homotopy groups of a rigid analytic DM stack $\mathscr{X}$ are defined to be those of $\mathscr{X}^{\mathrm{top}}$. So it seems likely that there is some natural functor **AlgDM → Rigid → Top**, which is well-defined in the sense of respecting the coarse space constructions. Naively, we also expect the statements of [BN05, Lemma 4.1, Theorem 4.2 and Theorem 4.3] hold verbatim when we replace "analytic" stacks by "rigid analytic."

Likewise, we might hope for a theory of uniformization of rigid analytic stacks similar to the corresponding theory for analytic stacks in the sense of the following statement. Note that homotopy groups of analytic and algebraic stacks $\mathscr{X}$ are defined to be those of $\mathscr{X}^{\mathrm{top}}$. The fundamental group of a topological stack $\mathscr{X}^{\mathrm{top}}$ classifies the connected covering spaces of $\mathscr{X}^{\mathrm{top}}$ because the pointed connected covering spaces of $\mathscr{X}^{\mathrm{top}}$ are in bijection with subgroups of $\pi_1(\mathscr{X}^{\mathrm{top}}, x)$. Recall that a **covering space** of $\mathscr{X}^{\mathrm{top}}$ is a representable map of stacks $\mathscr{Y} \to \mathscr{X}^{\mathrm{top}}$ that is stable under base change. In particular $\mathscr{X}^{\mathrm{top}}$ has a unique universal cover $\tilde{\mathscr{X}}$ up to equivalence. Accordingly, we say that a rigid analytic stack $\mathscr{X}$ is **simply connected** if $\pi_1(\mathscr{X}^{\mathrm{top}}, x)$ is trivial, where $\mathscr{X}^{\mathrm{top}}$ is the topological stack associated with $\mathscr{X}$.

Remark 6.3.8. *Since we have a theory of uniformization of analytic stacky curves, it seems worth considering whether something like the following statement can be proven:*

Every rigid analytic Deligne-Mumford stacky curve has a universal cover which is a simply connected rigid analytic DM stacky curve. All simply connected rigid analytic DM stacky curves have form similar to $C - K_\infty$: generally C with some compact subspace removed. That is, we can classify these rigid stacky curves according to their simply connected universal covers.

Remark 6.3.9. *If some statement like Remark 6.3.8 is true, or can be shown to be true, then it seems likely that something like the following statement will be true:*

Every uniformizable DM rigid analytic stacky curve $\mathscr{X}$ is the stack quotient $[X/G]$ of a rigid analytic space X by a finite group G.

Remark 6.3.10. *All of the "conjectures" stated in this section are versions of Theorems in [BN05] which hold for "ordinary" analytic stacks. It seems possible then that similar proofs, but using rigid analytic notions of convergence for example, would prove these statements. In particular though, we do not rely on any of these ideas in our proofs and want to emphasize these questions to indicate directions for future study. It is even fair to call these questions ill-defined, but taken together all we mean to do by including these statements is ask: "Can we do [BN05] for rigid analytic stacks?"*

6.4 Cusps of Drinfeld Modular Curves

In this section we consider the Drinfeld setting version of [DS05, Chapter 2, Sections 2 and 4]. Before we begin thinking about cusps, let us consider charts on the affine Drinfeld modular curve $Y_{\mathrm{GL}_2(A)}$ with $Y_{\mathrm{GL}_2(A)}(C) = Y^{\mathrm{an}}_{\mathrm{GL}_2(A)} = \mathrm{GL}_2(A)\backslash\Omega$.

6.4.1 Basics of Charts on Drinfeld Modular Curves

Let $\pi : \Omega \to Y^{\mathrm{an}}_{\mathrm{GL}_2(A)}$ be the quotient map, and let $\tau \in \Omega$. First we will define some coordinates. Let $\delta_\tau \overset{def}{=} \left(\begin{smallmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{smallmatrix}\right) \in \mathrm{GL}_2(C)$, where by $\overline{\tau}$ we denote our version of "complex conjugate" in C. That is, if $\tau = \sum_{i\in\mathbb{Z}} a_i T^i \in \Omega$, then recall that τ is "convergent" in the sense that $\lim_i |a_i|||T^i|| = 0$, where for $a \neq 0 \in A$ we have

$|a| \overset{def}{=} q^{\deg a}$, so $|a_i| = q^0$ for $a_i \in \mathbb{F}_q$ and $||T^i|| = q^i$. Fix $\varepsilon > 0$ and let $N \in \mathbb{Z}_{<0}$ be such that for $n < N$ we have $|a_n T^n| < \varepsilon$. Then $\overline{\tau} = \sum_{i \in \mathbb{Z}} b_i T^i \in \Omega$ is the unique element such that

$$\sum_{i+j=m} a_i b_j = 0$$

for all $m < N$ and for all $m \in \mathbb{Z}_{\geqslant N}$ the remaining coefficients in the series $\overline{\tau}$ have $b_m \overset{def}{=} a_m$. Define the **period of** τ by the integer

$$h_\tau \overset{def}{=} \#\left(Z(\mathrm{GL}_2(A))\,\mathrm{Stab}_\Gamma(\tau)/Z(\mathrm{GL}_2(A))\right) = \begin{cases} \frac{\#(\mathrm{Stab}_\Gamma(\tau))}{2}, & -\mathrm{Id} \in \mathrm{Stab}_\Gamma(\tau) \\ \#\,\mathrm{Stab}_\Gamma(\tau), & -\mathrm{Id} \notin \mathrm{Stab}_\Gamma(\tau). \end{cases}$$

The δ_τ maps $\tau \mapsto 0$ and $\overline{\tau} \mapsto \infty$. The isotropy subgroup of 0 is

$$\mathrm{Stab}_{\delta_\tau Z(\mathrm{GL}_2(A))\Gamma\delta_\tau^{-1}}(0)/Z(\mathrm{GL}_2(A)) = \left(\delta_\tau Z(\mathrm{GL}_2(A))\Gamma\delta_\tau^{-1}\right)_0/Z(\mathrm{GL}_2(A)),$$

which is conjugate to the isotropy subgroup of τ:

$$\delta_\tau\left(Z(\mathrm{GL}_2(A))\,\mathrm{Stab}_\Gamma(\tau)/Z(\mathrm{GL}_2(A))\right)\delta_\tau^{-1},$$

so is cyclic and of order h_τ. This group fixes 0 and ∞, and it consists of maps $z \mapsto az$, so those maps must be rotations through multiples of $\overline{\pi}/h_\tau$ about the origin because the group is finite cyclic. That is, δ_τ "straightens" neighborhoods of τ to neighborhoods of the origin since $\mathrm{GL}_2(A)$-equivalent points will be spaced apart by fixed angles. The coordinate neighborhood of $\pi(\tau)$ in $Y^{\mathrm{an}}_{\mathrm{GL}_2(A)}$ is the π-image of an the intersection of

1. a circular sector through $\overline{\pi}/h_\tau$ about τ in Ω, and

2. admissible opens sets in a pure cover of Ω (see Example 5.1.6).

So the identifying action of π is basically an "unwrapping" action of the h_τ power map taking a sector to the disk. We mention one last point for our intuition and notation before making the discussion above precise.

Corollary 6.4.1 ([DS05, 2.2.3]). *Let $\Gamma \leqslant \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Each $\tau \in \mathcal{H}$ has a neighborhood $U \subset \mathcal{H}$ such that for all $\gamma \in \Gamma$, if $\gamma(U) \cap U \neq \varnothing$ then $\gamma \in \Gamma_\tau$. Such neighborhoods have no elliptic points except possibly τ.*

In the Drinfeld setting, we expect cusps of a Drinfeld modular curve are stacky points, so they are elliptic points by Definition 6.2.7 which we use in [Fra23]. Now let us make things more precise.

Given $\pi(\tau) \in Y_{\mathrm{GL}_2(A)}$ and let U be some neighborhood in Ω the intersection of some neighborhood as in Corollary 6.4.1 and some admissible open from Example 5.1.6. Let $\psi : U \to C$ be the composite $\psi = \rho \circ \delta$, where $\delta = \delta_\tau$ and $\rho(z) = z^{h_\tau}$. Then $\psi(\tau') = (\delta(\tau'))^{h_\tau}$ is the straightening δ_τ followed by the h_τ-fold wrapping. See [DS05, Figure 2.2] for a sketch of the corresponding maps in $\mathbb{C}$.

Let $V = \psi(U) \subset C$. We know V is open by the open mapping theorem, indeed open sets for the rigid analytic topology are both open and closed in the metric topology. Since the projection π and the map ψ identify the same points of U, there should be an equivalence between $\pi(U)$ and $\psi(U)$, which we verify next. For any

$\tau_1, \tau_2 \in U$ we have

$$\begin{aligned}
\pi(\tau_1) = \pi(\tau_2) &\iff \tau_1 \in \mathrm{GL}_2(A)_{\tau_2} \\
&\iff \tau_1 \in \mathrm{GL}_2(A)_\tau \tau_2 \\
&\iff \delta(\tau_1) \in (\delta \mathrm{GL}_2(A)_\tau \delta^{-1})(\delta(\tau_2)) \\
&\iff \delta(\tau_1) = \mu_{h_\tau}^d(\delta(\tau_2)),
\end{aligned}$$

for some $d \in \mathbb{Z}$ and $\mu_{h_\tau} = e_A(z/h_\tau)$, where e_A is the exponential function from the definition of our parameter $u(z)$ at ∞ in Section 4.3, since $\delta \mathrm{GL}_2(A)_\tau \delta^{-1}$ is a cyclic h_τ-rotation group. So

$$\begin{aligned}
\pi(\tau_1) = \pi(\tau_2) &\iff \delta(\tau_1)^{h_\tau} = \delta(\tau_2)^{h_\tau} \\
&\iff \psi(\tau_1) = \psi(\tau_2).
\end{aligned}$$

This means there is an injective map $\varphi : \pi(U) \to V$ such that the following commutes:

$$\begin{tikzcd}
U \arrow[d, "\pi"] \arrow[dr, "\psi"] & \\
\pi(U) \arrow[r, "\varphi"] & V
\end{tikzcd}$$

and in fact since ψ is surjective by definition, we know φ is too.

It remains to check whether transition maps between coordinate charts are holomorphic (in the sense of [FvdP04, Definition 2.2.1]). Let $V_{12} \overset{def}{=} \varphi_1(\pi(U_1) \cap \pi(U_2))$

and let $V_{21} \overset{def}{=} \varphi_2(\pi(U_1) \cap \pi(U_2))$. Then consider the diagram

$$\begin{tikzcd} & \pi(U_1) \cap \pi(U_2) \arrow[dr, "\varphi_2"] & \\ V_{12} \arrow[ur, "\varphi_1^{-1}"] \arrow[rr, "\varphi_{12}"] & & V_{21} \end{tikzcd}$$

For $x \in \pi(U_1) \cap \pi(U_2)$ it suffices to check holomorphy in some neighborhood of $\varphi_1(x) \in V_{12}$. Suppose $x = \pi(\tau_1) = \pi(\tau_2)$ for some $\tau_1 \in U_2$ and $\tau_2 \in U_2$ with $\tau_2 = \gamma\tau_1$ for some $\gamma \in \mathrm{GL}_2(A)$. Let $U_{12} \overset{def}{=} U_1 \cap \gamma^{-1}(U_2)$. Then $\pi(U_{12})$ is a neighborhood of x in $\pi(U_1) \cap \pi(U_2)$ and $\varphi_1(\pi(U_{12}))$ is a neighborhood of $\varphi_1(x)$ in V_{12}.

If $\varphi_1(x) = 0$ then an input $u = \varphi_1(x')$ to φ_{21} in this neighborhood has form

$$u = \varphi_1(\pi(\tau')) = \psi_1(\tau') = \delta_1(\tau')^{h_1}$$

for some $\tau' \in U_{12}$ and h_1 the width of τ_1. Let $\tilde{\tau}_2 \in U_2$ be such that $\psi_2(\tilde{\tau}_2) = 0$ and the width is h_2. The corresponding output is

$$\begin{aligned} \varphi_2(x') &= \varphi_2(\pi(\gamma(\tau'))) \\ &= \psi_2(\gamma(\tau')) \\ &= \delta_2(\gamma(\tau'))^{h_2} \\ &= (\delta_2\gamma\delta_1^{-1})(\delta_1(\tau'))^{h_2} \\ &= (\delta_2\gamma\delta_1^{-1})(u^{1/h_1})^{h_2}. \end{aligned}$$

Then the only case where a transition map might not be holomorphic is when

$h_1 > 1$, and hence τ_1 and τ_2 are elliptic points. By construction U_2 contains at most one elliptic point and the local coordinate maps it to 0. If $h_1 > 1$ then τ_2 is the point $\tilde{\tau_2} \in U_2$ and the second straightening map is $\delta_2 = \delta_{\tau_2}$ and $h_2 = h_1$. That is, we have maps

$$0 \overset{\delta_1^{-1}}{\mapsto} \tau_1 \overset{\gamma}{\mapsto} \tau_2 \overset{\delta_2}{\mapsto} 0,$$

and

$$\infty \overset{\delta_1^{-1}}{\mapsto} \overline{\tau_1} \overset{\gamma}{\mapsto} \overline{\tau_2} \overset{\delta_2}{\mapsto} \infty,$$

so $\delta_2\gamma\delta_1^{-1} = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ for some $\alpha, \beta \in C^\times$. Then φ_{21} is the map

$$u \mapsto \left(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) u^{1/h}\right)^h = (\alpha/\beta)^h u,$$

which is holomorphic on $Y^{\mathrm{an}}_{\mathrm{GL}_2(A)} = \mathrm{GL}_2(A)\backslash\Omega$ since u has only a double pole at ∞.

6.4.2 Charts at Cusps

Now we will turn to $X^{\mathrm{an}}_\Gamma = \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$ for $\Gamma \leqslant \mathrm{GL}_2(A)$ a congruence subgroup containing the diagonal matrices in $\mathrm{GL}_2(A)$. For each $s \in \mathbb{P}^1(K)$, there is some $\delta = \delta_s \in GL_2(A)$ which maps $s \to \infty$. As in [Gek86, V.2.(2.5)], the stabilizer $\mathrm{Stab}_\Gamma(s)$ contains a maximal subgroup of form $\{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathfrak{B}\}$ for $\mathfrak{B}$ some fractional ideal of A. Since the exponential from Definition 4.3.1 is invariant under translations $z \mapsto z+b$ for $b \in A$, there is some $\mathfrak{B}$-stable, admissible subset Ω' of Ω such that u (from Definition 6.1.1) identifies $\mathfrak{B}\backslash\Omega'$ with a pointed neighborhood of 0 in C. For "sufficiently large" r we may use $\Omega' = \{z \in \Omega : |z|_i \geqslant r\}$, where $|\cdot|_i$ is the imaginary absolute value from [Gek86, V.1.(1.1)]. In general $\mathrm{Stab}_\Gamma(s)$ has form

$$\left\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : b \in \mathfrak{B},\ \text{and certain } a, d \in \mathbb{F}_q^\times\right\}$$

which contains transformations of form $z \mapsto \alpha z$. Let w denote the order of the cyclic group of these transformations. Then u^w is a local parameter at s.

We define the **width of a cusp** $s \in \mathbb{P}^1(K)$ to be

$$h_s = \#\left(\mathrm{Stab}_{\mathrm{GL}_2(A)}(\infty) / \mathrm{Stab}_{\delta Z(\mathrm{GL}_2(A))\Gamma\delta^{-1}}(\infty)\right).$$

See [DS05, Figure 2.6] for a sketch of the corresponding discussion for $\mathbb{C}$. At a cusp, infinitely many sectors in a given admissible open come together and the width of the cusp is the number of distinct strips up to isotropy. This is dual to the period of $\tau \in \Omega$ in that it is inversely proportional to the size of the isotropy subgroup. Since $\mathrm{GL}_2(A)_\infty$ is the set of all upper-triangular matrices, with the maximal subgroup

$$\mathrm{GL}_2(A)_\infty^u = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) : a \in A\}$$

and the subgroup of cyclic transformations $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ for $a, d \in \mathbb{F}_q^\times$, the group is infinite and cyclic, so width is characterized by

$$Z(\mathrm{GL}_2(A))(\delta\Gamma\delta^{-1})_\infty = Z(\mathrm{GL}_2(A))\langle\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)\rangle,$$

with $h > 0$, which is finite. We claim it is independent of choice of δ. Indeed

$$h_s = \#\left(\mathrm{GL}_2(A)_s / Z(\mathrm{GL}_2(A))\Gamma_s\right)$$

so width is well-defined on $X^{\mathrm{an}}_{\mathrm{GL}_2(A)}$ and is independent of δ. If $s \in \mathbb{P}^1(K_\infty)$ and $\gamma \in \mathrm{GL}_2(A)$ then

$$\begin{pmatrix} \text{width of } \gamma(s) \text{ under} \\ \gamma\Gamma\gamma^{-1} \end{pmatrix} = \begin{pmatrix} \text{width of } s \text{ under} \\ \Gamma \end{pmatrix}.$$

If Γ is normal in $\mathrm{GL}_2(A)$ then all cusps have the same wdith.

Let $U = U_s \stackrel{def}{=} \delta^{-1}(\mathscr{N} \cup \{\infty\})$, where $\mathscr{N} \stackrel{def}{=} \bigcup_{(n,x\in I)} D_{n,x}$, where $D_{n,x}$ are the affinoid spaces from Example 5.1.6 and I is the index set from the same Example. Let $\psi \stackrel{def}{=} \rho \circ \delta$, where $\rho(z) = u(z/h)$, where $u(z)$ is the parameter at ∞. Let $V \subset C$ be the image of ψ, i.e. $\psi : U \to V$ is the map $\tau \mapsto u(\delta(\tau)/h)$.

We see the same way as before that ψ and π do the same identifications about s. Once again, it remains to check that transition maps are holomorphic. Let $U_1 \subset \Omega$ correspond to $\delta_{\tau_1} \in \mathrm{GL}_2(C)$, where τ_1 has width h_1 and let $U_2 = \delta_2^{-1}(\mathscr{N} \cup \{\infty\})$. For each $x \in \pi(U_1) \cap \pi(U_2)$, we have $x = \pi(\tilde{\tau_1}) = \pi(\tau_2)$ for some $\tilde{\tau_1} \in U_1$ and $\tau_2 = \gamma(\tilde{\tau_1})$ for some $\gamma \in \Gamma$. Let $U_{12} = U_1 \cap \gamma^{-1}(U_2)$ be a neighborhood of $\tilde{\tau_1}$ in Ω. Then $\varphi_1(\pi(U_{12}))$ is a neighborhood of $\varphi_1(x)$ in $V_{12} = \varphi_1\left(\pi(U_1) \cap \pi(U_2)\right)$. If $h_1 > 1$ then $\tau_1 \notin U_{12}$, otherwise $\delta_2(\gamma(\tau_1)) \in \mathscr{N}$ is an elliptic point for Γ. In the classical proof, for $\mathscr{N}_2$ as in [DS05, Proof of Lemma 2.4.1], we know $\mathscr{N}_2$ does not contain any elliptic points for

$\mathrm{SL}_2(\mathbb{Z})$. However, $\mathscr{N}$ contains elliptic points for $\mathrm{GL}_2(A)$. So, if $h_1 > 1$ then we do not necessarily have that $0 \notin \varphi_1(\pi(U_{12}))$.

An input point $\varphi_1(x')$ to φ_{21} in V_{12} has form $q = (\delta_1(\tau'))^{h_1}$ and output

$$\begin{aligned}\varphi_2(x') &= \varphi_2(\pi(\gamma(\tau'))) \\ &= \psi_2(\gamma(\tau')) \\ &= u(\overline{\pi}\delta_2\gamma(\tau')/h) \\ &= u(\overline{\pi}\delta_2\gamma\delta_1^{-1}(u^{1/h_1})/h_2).\end{aligned}$$

Then as before the only case where a transition map might not be holomorphic is $h_1 > 1$ and $0 \in \varphi_1(\pi(U_{12}))$.

Let $U_1 \overset{def}{=} \delta_1^{-1}(\mathscr{N} \cup \{\infty\})$, where $\delta_1 : s_1 \mapsto \infty$ and let $U_2 \overset{def}{=} \delta_2^{-1}(\mathscr{N} \cup \{\infty\})$, where $\delta_2 : s_2 \mapsto \infty$. If $\pi(U_1) \cap \pi(U_2) \neq \varnothing$, then for some $\gamma \in \Gamma$ we have

$$\gamma\delta_1^{-1}(\mathscr{N} \cup \{\infty\}) \cap \delta_2^{-1}(\mathscr{N} \cup \{\infty\}) \neq \infty,$$

i.e. $\delta_2\gamma\delta_1^{-1}$ moves some point in $\mathscr{N} \cup \{\infty\}$ to another, so is a translation $\pm\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Then

$$\gamma(s_1) = \gamma\delta_1^{-1}(\infty) = \pm\delta_2^{-1}\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)(\infty) = s_2,$$

so $h_1 = h_2$ and the transition map moves a point in $\varphi_1(\pi(U_{12}))$, some $u = \psi_1(\tau) =$

$u(\overline{\pi}\delta_1(\tau)/h)$, to the point

$$\begin{aligned}\psi_2(\gamma(\tau)) &= u(\overline{\pi}\delta_2\gamma\delta_1^{-1}(\delta_1(\tau))/h)\\ &= u(\overline{\pi}((a/d)\delta_1(\tau)+b/d)/h)\\ &= u(\overline{\pi}(b/dh))u,\end{aligned}$$

which is holomorphic.

6.4.3 Isotropy Groups of Cusps

As usual let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices of $\mathrm{GL}_2(A)$. We have seen that for any $s \in \mathbb{P}^1(K)$, the stabilizer $\mathrm{Stab}_\Gamma(s)$ consists of some upper-triangular matrices

$$\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, d \in \mathbb{F}_q^\times, b \in A\}.$$

This group is infinite, but we know that the stabilizer of any point on a tame Deligne-Mumford stack is a finite cyclic group. To each s we associate a finite integer h_s its width, defined by

$$h_s = \#\left(\mathrm{Stab}_{\mathrm{GL}_2(A)}(s)/(Z(GL_2(A))\,\mathrm{Stab}_\Gamma(s))\right).$$

The order w of the cyclic group of transformations $z \mapsto \alpha z$ induced by some $\gamma \in \mathrm{Stab}_\Gamma(s)$ is h_s when $\Gamma \leqslant \mathrm{GL}_2(A)$ is a normal subgroup so that all cusps of X_Γ^{an} have the same width.

We need to think of some subgroup of the isotropy group of a cusp as the "right" automorphisms of that cusp to have a well-defined Deligne-Mumford stacky curve as our stacky Drinfeld modular curve, and the width gives us a natural-looking order of this group. The question becomes: "why can we ignore the infinite group of translations $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$ in the isotropy group?"

Recall from [Gek86, V.2.(2.4)] that the cusps of Γ, the orbits $\Gamma\backslash\mathbb{P}^1(K)$, are also described by {classes of ends in $\Gamma\backslash\mathscr{T}$} where $\mathscr{T}$ is the Bruhat-Tits tree of $\mathrm{PGL}_2(K_\infty)$ (from Section 4.2). These classes of ends are infinite graphs of the form within some

$$\bullet - \bullet - \bullet \to \cdots$$

quotient graph $\Gamma\backslash\mathscr{T}$. The translations $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)$ are clearly automorphisms of such ends in the sense of picking some point or another along the end to be the representative for a cusp. However, these automorphisms do not distinguish one class of an end from another, instead they describe the *class* of the end itself. That is, translations identify an entire half-line in $\Gamma\backslash\mathscr{T}$ with a single cusp Γs for some $s \in \mathbb{P}^1(K)$.

The key to this confusion comes from thinking of a given compact rigid analytic Drinfeld modular curve $X_\Gamma^{\mathrm{an}} = \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$ as a compactification of $Y_\Gamma^{\mathrm{an}} = \Gamma\backslash\mathscr{F}$, where $\mathscr{F}$ denotes the *fundamental domain for* Ω that we sketched an example of in Section 4.2. So far we have written $Y_\Gamma^{\mathrm{an}} = \Gamma\backslash\Omega$, but every point in Ω is $\mathrm{GL}_2(A)$-equivalent to some point in $\mathscr{F}$, and we find it more convenient to phrase our compactification in terms of the quotient of this fundamental domain rather than the full Ω since then the translations have "already disappeared."

To make sure we have a well-defined compactification which leaves the "right" size isotropy groups of cusps we will adopt the notation of [Vin12, Section 3.1.4]. Let

$$\Omega_c = \{z \in \Omega : \inf_{x \in K_\infty} |z - x| \geqslant c\}$$

denote a neighborhood of ∞ in Ω. Then the u function (recall 6.1.1) identifies $A \backslash \Omega_c$ with a pointed ball $B_r \backslash \{0\}$ for some small radius r. There exists some constant c_0 such that for all $c \geqslant c_0$ and $\gamma \in \mathrm{GL}_2(A)$, if $\Omega_c \cap \gamma(\Omega_c) \neq \varnothing$ then in fact $\gamma \in \mathrm{Stab}_\Gamma(\infty)$. Then for such a c we have an open immersion of rigid analytic spaces

$$\mathrm{Stab}_\Gamma(\infty) \backslash \Omega_c \hookrightarrow \mathrm{GL}_2(A) \backslash \Omega$$

given by the identification

$$\begin{aligned} \mathrm{Stab}_\Gamma(\infty) \backslash \Omega_c &\xrightarrow{\sim} B_{r^w} \backslash \{0\} \\ z &\mapsto u^w(z), \end{aligned}$$

where w is the order of the finite cyclic group of transformations $z \mapsto \alpha z$ contained in $\mathrm{Stab}_\Gamma(s)$, i.e. w is the width h_s of s. Note that when $\Gamma = \mathrm{GL}_2(A)$ itself, $w = q - 1$ as in [Vin12, Section 3.1.4] or [Gek86, V.2.(2.5)].

The upshot of compactifying Y_Γ^{an} according to this specification of a chart at ∞ is that thinking of compactifying a quotient of the fundamental domain $\mathscr{F}$ for Ω, as opposed to compactifying Ω and then taking a quotient, is we have already removed translations from the isotropy groups of cusps. Then for any given cusp $s \in \Gamma \backslash \mathbb{P}^1(K)$

we say

$$\mathrm{Aut}(s) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) : a, d \in \mathbb{F}_q^\times\}.$$

6.5 Moduli Interpretation

In this section we discuss moduli interpretations of Drinfeld modular curves for several kinds of congruence subgroup. We begin by recalling some analogies from the classical setting and some results from [Bre16] about the group $\mathrm{GL}_2(A)_2$ in particular. We can state moduli interpretations for several distinguished congruence subgroups Γ and their square-determinant subgroups Γ_2. Thanks to Mihran Papikian we also discuss some moduli interpretations for $\mathrm{SL}_2(A)$ and extend our ideas to its congruence subgroups.

First we recall the arithmetic congruence subgroups $\Gamma(N) \leqslant \Gamma_1(N) \leqslant \Gamma_0(N)$ of $\mathrm{GL}_2(A)$ containing the matrices

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right), \text{ and } \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N$$

respectively. In order to not go too far afield describing the degeneration of rank 2 Drinfeld modules in a moduli stack which is similar to the singular elliptic curves appearing as cusps of a classical modular curve, we content ourselves to consider rigid analytic affine coarse spaces. An excellent discussion of the elliptic curve version of the problem we are avoiding with this simplification is covered by [Alp23, Exercise 2.4.12.(b)] and [Alp23, Exercise 3.1.17]. As in the classical case, we have the following moduli interpretations

$$
\begin{aligned}
Y^{\mathrm{an}}_{\Gamma(N)} &- (\text{the moduli of rank 2 Drinfeld modules with a basis for } N \text{ torsion})\,, \\
Y^{\mathrm{an}}_{\Gamma_1(N)} &- (\text{the moduli of rank 2 Drinfeld modules with an } N\text{-torsion point})\,, \text{ and} \\
Y^{\mathrm{an}}_{\Gamma_0(N)} &- (\text{the moduli of rank 2 Drinfeld modules with an } N\text{-torsion group})\,.
\end{aligned}
$$

We next recall (Section 4.3) that the determinant of a rank 2 Drinfeld module $\varphi^z(T) = TX + g(z)X^q + \Delta(z)X^{q^2}$ is the rank 1 Drinfeld module

$$
\psi^z(T) \overset{def}{=} TX - \Delta(z)X^q.
$$

There is a Weil pairing

$$
w_T : \varphi[T] \times \varphi[T] \to \psi[T]
$$

sending $(x, y) \mapsto xy^q - x^q y$, the Moore determinant. From Gekeler we know $h(z)^{q-1} = -\Delta(z)$ and from [Bre16, (4.2)] we have

$$
\psi^z(T)(\lambda_T h(z)^{-1}) = 0,
$$

where $\lambda_T = \overline{\pi} e_A(T^{-1}) \in \rho[T]$, for $\rho = TX + X^q$ the Carlitz module. Then the determinant of φ^z is isomorphic to the Carlitz module via

$$
\psi^z = h(z)^{-1} \rho h(z)
$$

and is associated to the lattice $L_z \overset{def}{=} \overline{\pi} h(z)^{-1} A$. The N-torsion module of ψ^z is generated by $\overline{\pi} e_A(\frac{1}{a}) h(z)^{-1}$. Recall that the lattice $\overline{\pi} A$ associated to the Carlitz module ρ

relies on the choice $\overline{\pi} \in C$, which is defined up to a factor of $\mathbb{F}_q^\times$ and is transcendental over K. If we compare with this situation with elliptic curves, a given elliptic curve E defined by the lattice quotient $\mathbb{C}/(z\mathbb{Z}+\mathbb{Z})$ has

$$w_N\left(N^{-1}(u_1z+u_2), N^{-1}(v_1z+v_2)\right) = e^{2\pi i N^{-1}(u_1v_2-u_2v_1)},$$

where the right hand side does not depend on z. There is only one multiplicative group scheme $\mathbb{G}_m$ here, whereas there are many rank 1 Drinfeld modules, so $h(z)^{-1}$ serves to pick out the correct one ψ^z.

Breuer explains in [Bre16, Section 5] the following moduli intepretation:

$$Y^{\mathrm{an}}_{\mathrm{GL}_2(A)_2} - \begin{pmatrix} \text{the moduli of rank 2 Drinfeld modules} \\ \text{with } (\mathbb{F}_q^\times)^2\text{-classes of } T\text{-torsion} \\ \text{on their determinant modules} \end{pmatrix}.$$

Now we are in a position to discuss some new results. Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup and let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$. Then it is clear that $\Gamma_2 = \Gamma \cap \mathrm{GL}_2(A)_2$, so we state the folowing.

Proposition 6.5.1. *As stacky curves* $\mathscr{X}_{\Gamma_2} = \mathscr{X}_\Gamma \times_{\mathscr{X}_{\mathrm{GL}_2(A)}} \mathscr{X}_{\mathrm{GL}_2(A)_2}$.

Proof. As Γ and $\mathrm{GL}_2(A)_2$ are subgroups of $\mathrm{GL}_2(A)$ there are covers

$$\begin{array}{ccc} & & \mathscr{X}_{\mathrm{GL}_2(A)_2} \\ & & \downarrow \\ \mathscr{X}_\Gamma & \longrightarrow & \mathscr{X}_{\mathrm{GL}_2(A)} \end{array}$$

so that by the universal property of fiber-products we have

$$\begin{tikzcd}
\mathscr{X}_{\Gamma_2} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathscr{X}_{\Gamma} \times_{\mathscr{X}_{\mathrm{GL}_2(A)}} \mathscr{X}_{\mathrm{GL}_2(A)_2} \arrow[r] \arrow[d] & \mathscr{X}_{\mathrm{GL}_2(A)_2} \arrow[d] \\
& \mathscr{X}_{\Gamma} \arrow[r] & \mathscr{X}_{\mathrm{GL}_2(A)}
\end{tikzcd}$$

and in particular we claim the dotted arrow is an isomorphism. Consider all of the stacky curves as closed subspaces of some rigid analytic projective space, each having been canonically embedded so that the embedded curves are isomorphic to the stacky modular curves. Then as sets and in particular closed subspaces of a rigid analytic projective space we have

$$\mathscr{X}_{\Gamma_2}^{\mathrm{an}} = \mathscr{X}_{\Gamma}^{\mathrm{an}} \times_{\mathscr{X}_{\mathrm{GL}_2(A)}^{\mathrm{an}}} \mathscr{X}_{\mathrm{GL}_2(A)_2}^{\mathrm{an}}$$

since

$$\Gamma_2 \backslash (\Omega \cup \mathbb{P}^1(K)) = \Gamma \backslash (\Omega \cup \mathbb{P}^1(K)) \times_{\mathrm{GL}_2(A) \backslash (\Omega \cup \mathbb{P}^1(K))} \mathrm{GL}_2(A)_2 \backslash (\Omega \cup \mathbb{P}^1(K)).$$

By Rigid GAGA for stacks Lemma 5.5.4 the result follows for algebraic stacks. $\square$

Now we can give some new moduli interpretations as an easy consequence. Once again though, we consider only the coarse space when doing so.

$$Y^{\mathrm{an}}_{\Gamma(N)_2} - \left(\begin{array}{c}\text{the moduli of rank 2 Drinfeld modules with a basis for } N \text{ torsion,}\\ \text{with } (\mathbb{F}_q^\times)^2\text{-classes of } T\text{-torsion on their determinant,}\end{array}\right)$$

$$Y^{\mathrm{an}}_{\Gamma_1(N)_2} - \left(\begin{array}{c}\text{the moduli of rank 2 Drinfeld modules with an } N\text{-torsion point}\\ \text{with } (\mathbb{F}_q^\times)^2\text{-classes of } T\text{-torsion on their determinant, and}\end{array}\right)$$

$$Y^{\mathrm{an}}_{\Gamma_0(N)_2} - \left(\begin{array}{c}\text{the moduli of rank 2 Drinfeld modules with an } N\text{-torsion group}\\ \text{with } (\mathbb{F}_q^\times)^2\text{-classes of } T\text{-torsion on their determinant module.}\end{array}\right)$$

Next we consider the moduli interpretation for $\mathrm{SL}_2(A)$ itself, and for congruence subgroups of $\mathrm{SL}_2(A)$, the "Schottky groups," from [GvdP80]. Thanks to Mihran Papikian we have the following elegant formulation.

Let the pair (φ, α) denote a rank 2 Drinfeld module φ and $\alpha \in \mathbb{F}_q^\times = \mathrm{Aut}(\varphi)$. (Recall [Pap23, Definition 3.3.1] or Section 4.3 for automorphisms of a Drinfeld module). We say (φ, α) is **positively oriented** if $\alpha \in (\mathbb{F}_q^\times)^2$ and **negatively oriented** otherwise. Furthermore, we define $(\varphi, \alpha) \cong (\psi, \beta)$ if there exists some isomorphism of Drinfeld modules $u : \varphi \to \psi$ which preserves the sign of the orientation, i.e. $u\alpha = \beta$ with both of $\alpha, \beta \in (\mathbb{F}_q^\times)$ or neither α nor β a square. Then $\mathrm{SL}_2(A)\backslash\Omega$ classifies pairs $(\varphi, \alpha)/\cong$ (up to the isomorphism specified). That is,

$$Y^{\mathrm{an}}_{\mathrm{SL}_2(A)} - \left(\text{ the moduli of rank 2 oriented Drinfeld modules. }\right)$$

It should not be difficult to extend this moduli interpretation to the distinguished congruence subgroups $\Gamma^1(N)$, $\Gamma^1_1(N)$ and $\Gamma^1_0(N)$ of $\mathrm{SL}_2(A)$, where the superscript denotes the subgroups of the arithmetic congruence subgroups (from the beginning of this section) consisting of matrices with determinant 1. It is interesting to consider similar moduli interpretations for congruence subgroups Γ' containing $\Gamma_1 = \{\gamma \in \Gamma : \det \gamma = 1\}$. Clearly each of these will be some moduli of rank 2 Drinfeld modules with $(\mathbb{F}_q^\times)^2$-classes of T-torsion on their determinant in addition to some further torsion on their determinant.

Chapter 7

Geometry of Drinfeld Modular Forms

In this chapter we state and prove our main results about Drinfeld modular forms. We will see that like modular forms over $\mathbb{C}$ or a number field, Drinfeld modular forms are sections of some line bundle on a particular Drinfeld modular curve. We can compare Drinfeld modular curves for certain related pairs of subgroups of $\mathrm{GL}_2(A)$ and therefore compare their respective algebras of Drinfeld modular forms. Our first main result however is the most important, since with it we now have a technique to describe every algebra of Drinfeld modular forms currently in the literature and hopefully soon many more such algebras, by computing canonical rings of log stacky curves.

7.1 Drinfeld Modular Forms as Differentials

We prove our first main result:

Theorem 7.1.1 ([Fra23, Theorem 6.1])**.** *Let q be an odd prime and let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the scalar matrices of $\mathrm{GL}_2(A)$ and such that $\det(\gamma) \in (\mathbb{F}_q^\times)^2$ for every $\gamma \in \Gamma$. Let Δ be the divisor supported at the cusps of the modular curve $\mathscr{X}_\Gamma$ with rigid analytic coarse space $X_\Gamma^{an} = \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$. There is an isomorphism of graded rings*

$$M(\Gamma) \cong R(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(2\Delta)),$$

where $\Omega^1_{\mathscr{X}_\Gamma}$ is the sheaf of differentials on $\mathscr{X}_\Gamma$. The isomorphism of algebras is given by the isomorphisms of components $M_{k,l}(\Gamma) \to H^0(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(2\Delta)^{\otimes k/2})$ given by $f \mapsto f(dz)^{\otimes k/2}$.

Proof. Suppose $f \in M_{k,l}(\Gamma)$. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$f(\gamma z)d(\gamma z)^{\otimes k/2} = (cz+d)^k(\det\gamma)^{-l}\frac{\det\gamma^{k/2}}{(cz+d)^k}f(z)dz^{\otimes k/2},$$

where $k \equiv 2l \pmod{\frac{q-1}{2}}$. All of the factors of automorphy cancel and

$$f(\gamma z)d(\gamma z)^{\otimes k/2} = f(z)dz^{\otimes k/2},$$

so the differential form $f(dz)^{\otimes k/2} \in H^0(\Omega, \Omega_\Omega^{\otimes k/2})$ on the upper half-plane Ω is Γ-invariant. As in [GR96, Section (2.10)], we know $f(dz)^{\otimes k/2}$ is holomorphic on $\Gamma\backslash\Omega$. Since $\dfrac{de_A(z)}{dz} = 1$, we have $\dfrac{du}{u^2} = -\overline{\pi}dz$, so the differential dz in this case has a double pole at ∞. Then since f is holomorphic at the cusps of Γ,

$$\mathrm{div}(f(dz)^{\otimes k/2}) + k\Delta \geqslant 0,$$

and therefore $f(dz)^{\otimes k/2}$ is a global section of the twist by 2Δ of sheaf of holomorphic differentials on the rigid analytic space $X_\Gamma^{\mathrm{an}} = \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$. We claim this is a global section of (a twist by 2Δ of) the sheaf of differentials on the algebraic stack $\mathscr{X}$.

By rigid analytic GAGA, [FvdP04, Theorem 4.10.5], we know that the categories of coherent sheaves on the rigid space $\mathbb{P}_C^{n,\mathrm{an}}$ and coherent sheaves on $\mathbb{P}_C^n$ are equivalent and every closed analytic subspace of $\mathbb{P}_C^{n,\mathrm{an}}$ is the analytification of some closed subspace of $\mathbb{P}_C^n$. So, the sheaf $\Omega^1_{X_\Gamma^{\mathrm{an}}}(2\Delta)$ corresponds to the sheaf $\Omega^1_{X_\Gamma}(2\Delta)$ on the algebraic curve X_Γ which is the coarse space of $\mathscr{X}$. Finally, by [PY16, Theorem 7.4], we know the sheaves $\Omega^1_{\mathscr{X}_\Gamma^{\mathrm{an}}}(2\Delta)$ and $\Omega^1_{\mathscr{X}_\Gamma}(2\Delta)$ on the rigid analytic stacky curve and algebraic stacky curves $\mathscr{X}_\Gamma^{\mathrm{an}}$ and $\mathscr{X}_\Gamma$ respectively are equivalent. $\square$

7.2 Algebras of Drinfeld Modular Forms

7.2.1 A Special Case

When we compare the algebras of Drinfeld modular forms for Γ a congruence subgroup and $\Gamma_2 \leqslant \Gamma$ we arrive at the following conclusion.

Theorem 7.2.1 ([Fra23, Theorem 6.2]). *Let q be a power of an odd prime. Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices in $\mathrm{GL}_2(A)$. Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$. Then $M(\Gamma) \cong M(\Gamma_2)$, with*

$$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$

on each graded piece, where l_1, l_2 are the two solutions to $k \equiv 2l \pmod{q-1}$.

Remark 7.2.2. *Let $\mathscr{X}_\Gamma$ and $\mathscr{X}_{\Gamma_2}$ be the (stacky) Drinfeld modular curves whose coarse spaces are the algebraizations of $\Gamma\backslash(\Omega \cup \mathbb{P}^1(\mathbb{F}_q(T)))$ and $\Gamma_2\backslash(\Omega \cup \mathbb{P}^1(\mathbb{F}_q(T)))$ respectively. Let $D = K_{\mathscr{X}_\Gamma} + \Delta \sim K_{X_\Gamma} + R + \Delta$ and $D_2 = K_{\mathscr{X}_{\Gamma_2}} + \Delta_2 \sim K_{X_{\Gamma_2}} + R_2 + \Delta_2$ be log canonical divisors on $\mathscr{X}_{\Gamma_2}$ and $\mathscr{X}_{\Gamma_2}$, where K_{X_Γ} and $K_{X_{\Gamma_2}}$ are canonical divisors for the coarse spaces of $\mathscr{X}_\Gamma$ and $\mathscr{X}_{\Gamma_2}$ respectively, and Δ and Δ_2 are the log divisors of $\mathscr{X}_\Gamma$ and $\mathscr{X}_{\Gamma_2}$ respectively.*

1. *Suppose Γ is "square." Then $M(\Gamma_2) = M(\Gamma)$, and $K_{\mathscr{X}_{\Gamma_2}} + \Delta_2 = K_{\mathscr{X}_\Gamma} + \Delta$.*

2. *Suppose Γ is "non-square." Then each $s_2 \in \operatorname{supp}(\Delta_2)$ has $\#(\Gamma_2)_{s_2} = \frac{1}{2}(\#\Gamma_s)$ for any $s \in \operatorname{supp}(\Delta)$. If one could show that the cusps of $\mathscr{X}_\Gamma$ are in bijection with the cusps of $\mathscr{X}_{\Gamma_2}$ then the log canonical ring $R(\mathscr{X}_{\Gamma_2}; \Delta_2)$ is the spin canonical ring of the log curve $(\mathscr{X}_\Gamma; \Delta)$ as in [LRZ18a, Definition 2.9]. In the following proof we do not need such a bijection between cusps, and merely comment on this *hopefully* "easy" way to strengthen our result in future work.*

Remark 7.2.3. *Since there are many intermediate lemmata involved we break the proof of Theorem 7.2.1 up into the next few parts of this section. We state and prove the generalization afterwards.*

7.2.2 Properties of Γ_2

We begin with some group theory and elementary number theory which inspired our second main result and is instrumental in its proof.

Lemma 7.2.4. *Let $\Gamma \leqslant GL_2(A)$ be a congruence subgroup containing the diagonal matrices in $\mathrm{GL}_2(A)$. Let $\Gamma_2 = \{\gamma \in \Gamma : (\det \gamma) \in (\mathbb{F}_q^\times)^2\}$. Then Γ_2 is a normal subgroup* *of* Γ *with* $[\Gamma : \Gamma_2] = 2$, *and for any* $\alpha \in \mathbb{F}_q^\times \backslash (\mathbb{F}_q^\times)^2$, *the matrix* $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *is a representative for the unique non-trivial left coset of* Γ_2 *in* Γ.

Proof. Let $\varphi : \Gamma \to \mathbb{F}_q^\times$ be the map $\gamma \mapsto (\det \gamma)^{(q-1)/2}$. Then since $(\det \gamma)^{q-1} = 1$ for all $\gamma \in \Gamma$, we see $\ker \varphi = \Gamma_2$. If $\gamma \in \Gamma \backslash \Gamma_2$ then $(\det \gamma)^{(q-1)/2} \equiv -1 \pmod{q-1}$ so $\varphi(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$ as multiplicative groups and $[\Gamma : \Gamma_2] = 2$.

If $\gamma \in \Gamma \backslash \Gamma_2$, i.e. $\det(\gamma) \in \mathbb{F}_q^\times \backslash (\mathbb{F}_q^\times)^2$, then for any $\alpha \in \mathbb{F}_q^\times \backslash (\mathbb{F}_q^\times)^2$ there is some $\gamma_2 \in \Gamma_2$ with

$$\gamma = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) \gamma_2.$$

$\square$

We recall from elementary number theory the following.

Lemma 7.2.5. *Suppose* q *is odd. For a fixed* k, l *is such that* $k \equiv 2l \pmod{q-1}$, *if and only if*

$$l \equiv \begin{cases} \frac{k}{2} \pmod{q-1}, & \textit{or} \\ \frac{k}{2} + \frac{q-1}{2} \pmod{q-1}. \end{cases}$$

Proof. We know that $2l \equiv k \pmod{q-1}$ if and only if $2l - m(q-1) = k$ for some integer m. If $\gcd(2, q-1)$ does not divide k then there are no solutions, and if it does then there are exactly $\gcd(2, q-1) = 2$ distinct solutions modulo $q-1$. $\square$

To be explicit, we illustrate this Lemma with computations:

($\Rightarrow$) Suppose that $k = m(q-1) + 2l$ for some integer m. Since $q-1$ is even, k is even and $l = -m(\frac{q-1}{2}) + \frac{k}{2}$ so $l \equiv \frac{k}{2} \pmod{\frac{q-1}{2}}$. If m is even, $\frac{m}{2}$ is an integer, and otherwise $\frac{m-1}{2}$ is, so we have

$$l = \begin{cases} l_1 \equiv \frac{k}{2} \pmod{q-1}, & m \text{ even} \\ l_2 \equiv \frac{k}{2} + \frac{q-1}{2} \pmod{q-1}, & m \text{ odd.} \end{cases}$$

($\Leftarrow$) Suppose $l = l_1 \equiv \frac{k}{2} \pmod{q-1}$. Then $l_1 = n_1(q-1) + \frac{k}{2}$ for some n_1, so $k = -2n_1(q-1)+2l_1$. If $l = l_2 \equiv \frac{k}{2}+\frac{q-1}{2} \pmod{q-1}$ then $l_2 = n_2(q-1)+\frac{k}{2}+\frac{q-1}{2}$ for some n_2 and we have $k = -(2n_2+1)(q-1)+2l_2$. In either case we conclude that $k \equiv 2l \pmod{q-1}$.

7.2.3 Cusps and Elliptic Points

We wish to compare the cusps and elliptic points on the Drinfeld modular curves for Γ and Γ_2. Our notion of elliptic point is slightly different from Gekeler's so that it adapts to the notion of a stacky Drinfeld modular curve more naturally. So, we discuss some of the properties of our elliptic points with the next two group-theoretic results.

Lemma 7.2.6. *Let q be a power of an odd prime, let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices in $\mathrm{GL}_2(A)$. Suppose e_1 and e_2 are distinct elliptic points for Γ. Then the stabilizers Γ_{e_1} and Γ_{e_2} are $\mathrm{GL}_2(A)$-conjugate.*

Proof. Since both Γ_{e_1} and Γ_{e_2} stricty contain $\mathbb{F}_q^\times$ by definition of an elliptic point, and each stabilizer is a subgroup of $\mathrm{GL}_2(A)_{e_i}$, where $i = 1$ or 2, both elliptic points for Γ are also elliptic points on Ω. Then e_1 and e_2 are $\mathrm{GL}_2(A)$-equivalent to each other. If

$\gamma e_1 = e_2$ for $\gamma \in \mathrm{GL}_2(A)$ and $\gamma' \in \Gamma_{e_1}$, then

$$\begin{aligned}\gamma\gamma'\gamma^{-1}e_2 &= \gamma\gamma'\gamma^{-1}(\gamma e_1)\\ &= \gamma e_1\\ &= e_2.\end{aligned}$$

□

Lemma 7.2.7. *Let q be a power of an odd prime, let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices of* $\mathrm{GL}_2(A)$. *Let* $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\det\Gamma)^2\}$. *Let* $e \in \mathrm{Ell}(\Gamma_2)$. *Then*

$$[\Gamma_e : (\Gamma_2)_e] = \begin{cases} 1, & \text{if } \Gamma \text{ is ``square''} \\ 2, & \text{if } \Gamma \text{ is ``non-square.''} \end{cases}$$

Proof. By definition, the stabilizer Γ_e strictly contains $\mathbb{F}_q^\times$ and as this is a subgroup of the stabilizer $\mathrm{GL}_2(A)_e$, we see that e is an elliptic point for $\mathrm{GL}_2(A)$, i.e. an elliptic point on Ω, so we know $\mathrm{GL}_2(A)_e \cong \mathbb{F}_{q^2}^\times$. This means $(\Gamma_2)_e \trianglelefteq \Gamma_e \trianglelefteq \mathrm{GL}_2(A)_e \cong \mathbb{F}_{q^2}^\times$ and in particular since

$$(\Gamma_2)_e = \ker((\det)^{\frac{q-1}{2}} : \Gamma_e \to \mathbb{F}_q^\times),$$

the result is immediate according to whether $(\det)^{\frac{q-1}{2}}$ is surjective onto $\{\pm 1\}$. By Lemma 7.2.6, we need only check whether Γ_e contains some γ with $\det\gamma \in \mathbb{F}_q^\times \backslash (\mathbb{F}_q^\times)^2$ to determine the index of the stabilizer $(\Gamma_2)_e$ for all elliptic points e. □

The main idea for this step of the proof of Theorem 7.2.1 is the following comparison between elliptic points and cusps for Γ and Γ_2.

Proposition 7.2.8. *Let q be a power of an odd prime, let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices of $\mathrm{GL}_2(A)$. Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\det \Gamma)^2\}$.*

1. $\mathrm{Ell}(\Gamma) = \mathrm{Ell}(\Gamma_2)$,

2. $\mathcal{C}_\Gamma \subseteq \mathcal{C}_{\Gamma_2}$

Proof. Suppose $e_2 \in \mathrm{Ell}(\Gamma_2)$, so by definition the stabilizer $(\Gamma_2)_{e_2}$ is strictly larger than $\mathbb{F}_q^\times$. Since $(\Gamma_2)_{e_2}$ is a subgroup of Γ_{e_2}, it must be that Γ_{e_2} strictly contains $Z(\mathbb{F}_q)$, so $e_2 \in \mathrm{Ell}(\Gamma)$.

For the same reason, if $e \in \mathrm{Ell}(\Gamma)$, then e is an elliptic point on Ω, and we know $\mathrm{GL}_2(A)_e \cong \mathbb{F}_{q^2}^\times$. In particular, as $\mathbb{F}_{q^2}^\times$ and $\mathbb{F}_q^\times$ are cyclic groups, we know $(\Gamma_2)_e$ and Γ_e are cyclic and we have $1 \trianglelefteq Z(\mathbb{F}_q) \trianglelefteq (\Gamma_2)_e \trianglelefteq \Gamma_e \trianglelefteq \mathrm{GL}_2(A)_e \cong \mathbb{F}_{q^2}^\times$. Since $q-1 \mid \#\Gamma_e$, there is some $1 < n \leqslant q+1$ such that $n \mid q-1$ and $\#\Gamma_e = n(q-1)$. Suppose that $\langle \gamma \rangle = \Gamma_e$. Then the left cosets of $\mathbb{F}_q^\times$ in Γ_e have representatives

$$\gamma^j \begin{pmatrix} \frac{1}{\alpha_i} & 0 \\ 0 & \frac{1}{\alpha_i} \end{pmatrix}$$

for $1 \leqslant j \leqslant n(q-1)$ and $\alpha_i \in \mathbb{F}_q^\times$, so we can write

$$\Gamma_e/\mathbb{F}_q^\times \cong \mathbb{F}_q^\times \oplus \frac{\gamma}{\alpha_0}\mathbb{F}_q^\times \oplus \cdots \oplus \frac{\gamma}{\alpha_{q-1}}\mathbb{F}_q^\times \oplus \frac{\gamma^2}{\alpha_0}\mathbb{F}_q^\times \oplus \cdots \oplus \frac{\gamma^{(n-1)(q-1)}}{\alpha_{q-1}}\mathbb{F}_q^\times.$$

We claim that if Γ is "non-square," the cosets with representatives γ^j/α_i with j even form a subgroup isomorphic to $(\Gamma_2)_e/\mathbb{F}_q^\times$. If Γ is "non-square" then by Lemma 7.2.7 we know that Γ_e contains some γ' with $\det \gamma'$ a non-square, so $\det \gamma$ is non-square. Otherwise we would have $\gamma^n = \gamma'$ for some n, and so with $\det \gamma \in (\mathbb{F}_q^\times)^2$, we would have $\det \gamma'$ a square, which is a clear contradiction. Then for any even j we have

$$\det\left(\frac{\gamma^j}{\alpha}\right) = \frac{\det \gamma^j}{\alpha^2}$$

is a quotient of squares so is a square. For odd j, since $\det \gamma^j \in \mathbb{F}_q^\times \backslash (\mathbb{F}_q^\times)^2$ then for any $\alpha' \in \mathbb{F}_q^\times$ non-square, there is some $\gamma_2 \in \Gamma_2$ such that $\gamma = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) \gamma_2$. However, whether a given α_i is a square or not,

$$\det(\gamma^j/\alpha_i) = \frac{\alpha' \det \gamma_2}{\alpha_i^2},$$

which is not a square. Otherwise if Γ is "square," by Lemma 7.2.7 we have $\Gamma_e = (\Gamma_2)_e$.

Whether Γ is square or not, $(\Gamma_e)/\mathbb{F}_q^\times$ has a nontrivial subgroup isomorphic to $(\Gamma_2)_e/\mathbb{F}_q^\times$, so the stabilizer of e in Γ_2 strictly contains $\mathbb{F}_q^\times$ and $e \in \mathrm{Ell}(\Gamma_2)$.

Now we consider cusps. Let $s \in \mathbb{P}^1(K)$. Then $\Gamma s \supseteq \Gamma_2 s$, i.e. the action of Γ_2 partitions $\mathbb{P}^1(K)$ more finely than the action of Γ. If $s_1, \cdots, s_n$ are cusps of Γ, we write $\Gamma\backslash\mathbb{P}^1(K) = \Gamma s_1 \sqcup \cdots \sqcup \Gamma s_n$, and then

$$\Gamma s_i = \Gamma_2 s_i \sqcup (\Gamma\backslash\Gamma_2) s_i.$$

If the points of $\mathbb{P}^1(K)$ in the orbits $(\Gamma\backslash\Gamma_2)s_i$, under the action by Γ_2 have orbit representatives $t_1, \cdots, t_m$ then we can write

$$\Gamma_2\backslash\mathbb{P}^1(K) = \Gamma_2 s_1 \sqcup \cdots \sqcup \Gamma_2 s_n \sqcup \Gamma_2 t_1 \sqcup \cdots \sqcup \Gamma_2 t_m,$$

so the cusps of Γ_2 are $\mathcal{C}_{\Gamma_2} = \{s_1, \cdots, s_n, t_1, \cdots, t_m\}$, which contains $\mathcal{C}_\Gamma$. $\square$

Corollary 7.2.9. *Let q be a power of an odd prime. Let $\Gamma \leqslant GL_2(A)$ be a congruence subgroup. Let $\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$. Suppose that $\Gamma_1 \leqslant \Gamma' \leqslant \Gamma$ for some congruence subgroup Γ'. Then $\mathcal{C}_\Gamma \subseteq \mathcal{C}_{\Gamma'}$, i.e. the cusps of Γ are some subset of the cusps of Γ'*

Proof. The proof of the second part of Proposition 7.2.8 about the cusps did not make any particular use of the special choice of $\Gamma' = \Gamma_2$, and so holds in this more general situation. $\square$

7.2.4 Modularity and Series Expansions at Cusps

Our next steps in the proof of Theorem 7.2.1 deal with the u-series expansions of modular forms.

Proposition 7.2.10. *Let f be holomorphic on Ω and at the cusps of Γ_2, and let $\beta = \alpha^2 \in \mathbb{F}_q^\times$, where α generates $\mathbb{F}_q^\times$. Suppose that $f(\gamma z) = (\det \gamma)^{-l}(cz+d)^k f(z)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_2$. Then*

$$f\left(\left(\begin{smallmatrix} \beta & 0 \\ 0 & 1 \end{smallmatrix}\right) z\right) = f(\beta z) = \beta^{-k/2} f(z).$$

Proof. Since $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) \in \Gamma_2$ we have $f\left(\dfrac{\alpha z}{\alpha}\right) = f(z) = \alpha^{-2l}\alpha^k f(z)$, so $\alpha^{k-2l} = 1$.

Suppose that x generates the cyclic group $\mathbb{F}_q^\times$, so $\alpha = x^n$ for some n. If $\gcd(n, q-1) = 1$, i.e. α is a generator, then we claim $k \equiv 2l \pmod{q-1}$. The order of α is

$$\#\langle\alpha\rangle = \frac{q-1}{\gcd(n, q-1)} = q-1,$$

and we can write $k \equiv 2l \pmod{\#\langle\alpha\rangle}$ so

$$\gcd(n, q-1)(k-2l) \equiv 0 \pmod{q-1}.$$

But $\gcd(n, q-1)$ is coprime to $q-1$, from which the claim follows.

Since $\beta = \alpha^2$, we have $\beta^{2(k-2l)} = 1$, so

$$2k = 4l \pmod{q-1}.$$

Then we have $k \equiv 2l \pmod{\frac{q-1}{2}}$, since if $2k = m(q-1) + 4l$ for some m, we can write

$$k = m\left(\frac{q-1}{2}\right) + 2l.$$

Then $l \equiv \dfrac{k}{2} \pmod{\dfrac{q-1}{2}}$, so

$$f\left(\left(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\right) z\right) = f(\beta z) = \beta^{-k/2} f(z),$$

since $\left(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\right) \in \Gamma_2$. This matrix has square determinant by assumption and is in Γ by our assumption that Γ contains all matrices of form $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha'\end{smallmatrix}\right)$, for any $\alpha, \alpha' \in \mathbb{F}_q^\times$. $\square$

We complete the proof of Theorem 7.2.1 with the following result.

Proposition 7.2.11. *Suppose Γ is "non-square." Let f be a modular form of weight k and type l for Γ_2. Then there are two modular forms f_1 and f_2 for Γ of weight k and types $l_1 \equiv k/2 \pmod{q-1}$ and $l_2 \equiv k/2 + (q-1)/2 \pmod{q-1}$ respectively, such that $f = f_1 + f_2$.*

Proof. Suppose that $f(\gamma_2 z) = (\det \gamma_2)^{-l}(cz+d)^k f(z)$ for $\gamma_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_2$. Write the u-series $f(z) = \sum_{n \geqslant 0} a_n u^n$. Let $\beta = \alpha^2 \in \mathbb{F}_q^\times$, where α generates $\mathbb{F}_q^\times$. By Proposition 7.2.10, $f(\beta z) = \beta^{-k/2} f(z)$. Using this relationship, we have from Lemma 6.1.3

$$f(\beta z) = \sum_{n \geqslant 0} a_n \beta^{-n} u^n = \beta^{-k/2} \left(\sum_{n \geqslant 0} a_n u^n \right),$$

so for each non-zero a_n we have $\beta^{-n} = \beta^{-k/2}$ or $\alpha^{-2n} = \alpha^{-k}$. Then $k \equiv 2n \pmod{q-1}$, so by removing the zero summands from the u-series, we may write

$$f(z) = \sum_{n \equiv k/2 \pmod{q-1}} a_n u^n + \sum_{n \equiv k/2 + (q-1)/2 \pmod{q-1}} a_n u^n.$$

Let $\alpha \in \mathbb{F}_q^\times$ be some non-square, so by Lemma 6.1.3 we have $u(\alpha z) = \alpha^{-1} u(z)$. Let

$$f_1 = \sum_{n \equiv k/2 \pmod{q-1}} a_n u^n$$

and

$$f_2 = \sum_{n \equiv k/2 + (q-1)/2 \pmod{q-1}} a_n u^n$$

be the modular forms for Γ_2 uniquely determined by their u-series by Lemma 6.1.9. Then

$$f_1(\alpha z) = \sum_{n \equiv k/2 \pmod{q-1}} a_n \alpha^{-n} u^n = \alpha^{-l_1} \sum_{n \equiv k/2 \pmod{q-1}} a_n u^n,$$

where $l_1 \equiv \dfrac{k}{2} \pmod{q-1}$. Let $\gamma \in \Gamma \backslash \Gamma_2$. For any $\alpha \in \mathbb{F}_q^\times \backslash (\mathbb{F}_q^\times)^2$ there is some $\gamma_2 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_2$ such that

$$\gamma = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) \gamma_2,$$

so

$$f_1(\gamma z) = f_1(\alpha\gamma_2 z) = \alpha^{-l} f_1(\gamma_2 z) = \alpha^{-l}\det(\gamma_2)^{-l}(cz+d)^k f_1(z) = \det(\gamma)^{-l}(cz+d)^k f_1(z)$$

and f_1 is a modular form for Γ. Likewise we have

$$f_2(\alpha z) = \sum_{n\equiv k/2+(q-1)/2 \pmod{q-1}} a_n \alpha^{-n} u^n = \alpha^{-l_2} \sum_{n\equiv k/2+(q-1)/2 \pmod{q-1}} a_n u^n,$$

where now $l_2 \equiv \dfrac{k+q-1}{2} \pmod{q-1}$, so for γ, α and γ_2 as above,

$$f_2(\gamma z) = \alpha^{-l}\det(\gamma_2)^{-l}(cz+d)^k f_2(z)$$

and f_2 is a modular form for Γ. □

7.3 A Generalization

Gebhard Böckle suggested the following generalization of Theorem 7.2.1.

Theorem 7.3.1 ([Fra23, Theorem 6.12])**.** *Let q be a power of an odd prime. Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup. Let $\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$. Suppose that Γ' is such that $\Gamma_1 \leqslant \Gamma' \leqslant \Gamma$. Then as algebras*

$$M(\Gamma) = M(\Gamma'),$$

and each component $M_{k,l}(\Gamma')$ is some direct sum of components $M_{k,l'}(\Gamma)$ for some nontrivial l'.

Remark 7.3.2. *The subgroups* Γ' *which appear in the statement of Theorem 7.3.1 may be thought of as the inverse image under* $\det : \Gamma \to \mathbb{F}_q^\times$ *of some subgroup of* $\mathbb{F}_q^\times$.

Proof. (Theorem 7.3.1) Write $f|_\gamma$ for the (Peterson) **slash operator** of weight k and type l for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ defined by

$$f|_\gamma \overset{def}{=} \det(\gamma)^l (cz+d)^{-k} f(\gamma z).$$

If $f \in M_{k,l}(\Gamma')$, by normality $\Gamma' \trianglelefteq \Gamma$ we have that $f|_\gamma$ is weakly modular of weight k and type l for any $\gamma \in \Gamma$. Since the cusps of Γ are some subset of the cusps of Γ' we see that $f|_\gamma$ is holomorphic at the cusps of Γ since f is holomorphic at the cusps of Γ', indeed the u-series of expansions of $f|_\gamma$ and f agree at the cusps of Γ'.

The action of Γ' is trivial, so we have an action of the finite group

$$\Gamma/\Gamma' = \det(\Gamma)/\det(\Gamma'),$$

which has order some divisor of $q-1$ since $1 \leqslant \det \Gamma' \leqslant \det \Gamma \leqslant \mathbb{F}_q^\times$. Then we may describe the group ring $\mathbb{F}_q[\Gamma/\Gamma']$ via idempotents as follows. Let $n' \overset{def}{=} \#(\det \Gamma')$ and let $n \overset{def}{=} \#(\det \Gamma)$. Then

$$\mathbb{F}_q[\Gamma/\Gamma'] = \bigoplus_{i=0}^{n/n'-1} \mathbb{F}_q e_i,$$

where Γ acts on the e_i via maps $\gamma \mapsto (\det \gamma)^{in'}$. So as Γ-modules, we have

$$M_{k,l}(\Gamma') = \bigoplus_i M_{k,l}(\Gamma') e_i,$$

where

$$M_{k,l}(\Gamma')e_i = M_{k,l+in'}(\Gamma).$$

Finally, since modular forms for Γ' are holomorphic at the cusps of Γ', and since the cusps of Γ are a subset of the cusps of Γ', we know Γ'-modular forms are holomorphic at the cusps of Γ. $\square$

Remark 7.3.3. *One can verify that the slash operators $f|_\gamma$ are holomorphic at the cusps of Γ directly by considering their u-series expansions at small neighborhoods of the cusps of Γ.*

7.3.1 Slash Operators at Cusps

Write $f|_\gamma$ for the (Peterson) **slash operator** of weight k and type l for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ defined by

$$f|_\gamma \overset{def}{=} \det(\gamma)^l (cz+d)^{-k} f(\gamma z).$$

If $f \in M_{k,l}(\Gamma)$ and $\gamma \in \Gamma$, then

$$f|_\gamma = f.$$

Let $\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$. Suppose that $\Gamma_1 \leqslant \Gamma' \leqslant \Gamma$ for some congruence subgroup Γ', i.e. Γ' is the inverse image of some subgroup $G' \leqslant \mathbb{F}_q^\times$. We call such a Γ' a **Böckle subgroup**. Note that Böckle subgroups Γ' are all normal in Γ and in any Γ'' such that $\Gamma' \leqslant \Gamma'' \leqslant \Gamma$.

Proposition 7.3.4. *Fix some Böckle subgroup Γ' of a congruence subgroup $\Gamma \leqslant \mathrm{GL}_2(A)$. For any $f \in M_{k,l}(\Gamma')$ the slash operator $f|_\gamma$ is weakly modular of weight k*

and type l for all $\gamma \in \Gamma$. Furthermore, $f|_\gamma$ is holomorphic at the cusps of Γ.

Proof. The first part of the claim follows since $\Gamma' \trianglelefteq \Gamma$. We will verify explicitly the second part about holomorphy by considering u-series expansions of $f|_\gamma$ at the cusps of Γ', since as sets

$$\{\text{cusps of } \Gamma\} \subseteq \{\text{cusps of } \Gamma'\}.$$

Let $s = \gamma(\infty)$ for some $\gamma \in \Gamma$. The local map $\psi : U \to V$ around s has form $\psi = \rho \circ \delta$ of form $\tau \mapsto z \mapsto u$, where $\delta = \gamma^{-1}$, $\rho(z) = u(z/h)$ and h is the width of s. Let ω denote some holomorphic differential on $X_{\Gamma'}$. Since ω is holomorphic at the cusps of $X_{\Gamma'}$ the local differential $\omega|_V$ has the form $g(u)u(du)^n$ for some n, where g is holomorphic at 0. So on $U - \{s\}$, the form f is the pullback under ψ of $\omega|_{V-\{0\}}$ to $f(\tau)(d\tau)^n$. Whereas in the classical case (see [DS05, Page 80]) we have $f = \tilde{f}|_\delta$ of weight $2n$, where $\tilde{f}(z) = (2\pi i h^{-1})^n q^n g(q)^n$, for $q = \exp(2\pi i z/h)$, in the Drinfeld setting we have $f = \tilde{f}_\delta$ of weight k, where

$$\tilde{f}(z) = \left(\frac{\overline{\pi}}{h}\right) u^n g(u),$$

with $u = (\overline{\pi} e_A(z/h))^{-1}$. Then $f|_\gamma = \tilde{f}$ is holomorphic at the cusp ∞.

Now let $\pi(T) \in A$ be some monic polynomial and let p denote the ideal it generates. Let

$$W_p \overset{def}{=} \left(\begin{smallmatrix} 0 & -1 \\ \pi & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$$

be the matrix interchanging ∞ and 0. By composing the map W_p with $\gamma \in \Gamma$ and taking u-series expansions in small neighborhoods of each of the cusps of Γ' we see that $f|_\gamma$ is holomorphic at the cusps of Γ' in much the same way we verified at ∞. $\square$

Chapter 8

Computing Algebras of Drinfeld Modular Forms

In this chapter we discuss some applications and sample calculations using the theory of chapter 7. We also indicate how we intend to strengthen this theory in coming work and some open problems for further investgation.

8.1 An Algorithm

The program of this manuscript enables us to write an algorithm that takes on input some congruence subgroup of $\mathrm{GL}_2(A)$ and returns the algebra of Drinfeld modular forms for that subgroup.

0. **Set Up**

    - Fix q a power of an odd prime;

- Pick a congruence subgroup Γ (e.g. $\Gamma(N)$, $\Gamma_1(N)$ or $\Gamma_0(N)$, some Γ_2 within one of the previous, or some Γ' as in Theorem 7.3.1);

1. **Geometric Invariants**

   - **Genus** - Determine $g(\mathscr{X}_\Gamma)$ e.g. with [Gek01], [GvdP80] (for subgroups of $\mathrm{SL}_2(A)$ when N has odd degree) or Riemann-Hurwitz;
   - **Stacky Points**
     - **Elliptic points** - Determine whether Γ contains non-trivial stabilizers for the unique elliptic point of $\mathrm{GL}_2(A)$ in Ω;
     - **Cusps** - Consider the actions of Γ and $\Gamma_2 \leqslant \Gamma$ on $(A/N)^2_{\mathrm{prim}}$, the vectors in $(A/N) \times (A/N)$ which span a non-zero direct summand in a Chinese Remainder Theorem decomposition (up to $\mathbb{F}_q^\times$-scalars); or compare the quotient graphs $\Gamma\backslash\mathscr{T}$ and $\Gamma_2\backslash\mathscr{T}$, where $\mathscr{T}$ is the Burhat-Tits tree of $\mathrm{PGL}_2(K_\infty)$ using the algorithm of [GN95]
       (Note: in practice this is often difficult. We say more in Section 8.3.)

2. **Computing the Log Canonical Ring** - by either [VZB22] (see e.g. Figure 3.2), [O'D15], [CFO24], [LRZ16], or directly with e.g. Magma, compute as explicitly as possible the log canonical ring $R(\mathscr{X}_{\Gamma_2}, 2\Delta)$ using the invariants you have computed above;

3. **Recovering the Algebra** - by either Theorem 7.2.1 or Theorem 7.3.1, compare the algebra of modular forms for square determinant matrices Γ_2 with the chosen Γ to determine which generators are actually Γ-forms (as opposed to Γ_2-forms)

and pare down the relations from the previous step into those among these Γ-generators.

8.2 Known Examples

We consider several examples from the literature which we can now treat using the geometry introduced in Chapter 7.

8.2.1 Drinfeld modular forms for $\mathrm{GL}_2(A)$

Example 8.2.1. *Let $\mathscr{X}$ be the Drinfeld modular curve with coarse space X whose analytification is $X^{an} = \mathrm{GL}_2(A)_2 \backslash (\Omega \cup \mathbb{P}^1(K))$. Then $\mathscr{X}$ is a stacky $\mathbb{P}^1$ with two stacky points:*

- *a point P_e with a stabilizer of order $\dfrac{q+1}{2}$ corresponding to the unique elliptic point of Ω (note that $\mathrm{GL}_2(A)$ is "non-square")*
- *a cusp, denoted ∞, with a stabilizer of order $\dfrac{q-1}{2}$.*

Let

$$D = K_{\mathscr{X}} + 2\Delta \sim K_{\mathbb{P}^1} + \left(1 - \frac{1}{\frac{q+1}{2}}\right) P_e + \left(1 + \frac{1}{\frac{q-1}{2}}\right) \infty + 2\infty$$

be a log stacky canonical divisor on $\mathscr{X}$. Then by [O'D15, Theorem 6] we have

$$R_D \cong C[g, h] \cong M(\mathrm{GL}_2(A)_2).$$

8.2.2 Drinfeld modular forms for $\Gamma_0(T)$

We know from [Gek01, Theorem 8.1] genus formulae for the Drinfeld modular curves associated to $\Gamma(N), \Gamma_1(N)$ and $\Gamma_0(N)$. If $\deg N > 1$, then $g(X(N)) > 0$.

Consider $M(\Gamma_1(\alpha T+\beta))$ and $M(\Gamma_0(\alpha T+\beta))$. We know from [DK23, Theorem 4.4] that for R any ring such that $A \subset R \subset C$, the R-algebra of Drinfeld modular forms $M(\Gamma_0(T))_R$ is generated by $E_T(z)$ (from Example 6.1.12), and the Drinfeld modular forms

$$\Delta_T(z) \stackrel{def}{=} \frac{g(Tz) - g(z)}{T^q - T} \text{ and } \Delta_W(z) \stackrel{def}{=} \frac{T^q g(Tz) - Tg(z)}{T^q - T}$$

for $\Gamma_0(T)$ (from [DK23, Equation (4.1)]). Furthermore, [DK23, Theorem 4.4] tell us that the surjective map $R[U,V,Z] \to M(\Gamma_0(T))_R$ defined by $U \to \Delta_W$, $V \to \Delta_T$ and $Z \to E_T$ induces an isomorphism

$$R[U,V,Z]/(UV - Z^{q-1}) \cong M(\Gamma_0(T))_R.$$

Note that from [DK23, Proposition 4.3(3)] we know that $M_{k,l}(\Gamma_0(T))$ has an integral basis, i.e. a basis consisting of modular forms with coefficients in A.

Recall that from [DK23, Section 4] we know the only two cusps of $\Gamma_0(T)$, which we write 0 and ∞, are exchanged by the matrix

$$W_T \stackrel{def}{=} \left(\begin{smallmatrix} 0 & -1 \\ T & 0 \end{smallmatrix}\right).$$

We will use $M(\Gamma_0(T)) = C[U,V,Z]/(UV - Z^2)$ from [DK23, Theorem 4.4] to make sure that the log stacky canonical ring of the corresponding Drinfeld modular curve $\mathscr{X}_{\Gamma_0(T)_2}$ does in fact compute this algebra of Drinfeld modular forms for $\Gamma_0(T)_2$.

Example 8.2.2. *Since $UV - Z^2$ describes a conic, we know that the curve $C[U,V,Z]/(UV - Z^2) \subset \mathbb{P}^2_C$ is rational, and all rational curves have genus 0. There are 2 cusps, say 0 and ∞ for $\mathscr{X}_{\Gamma_0(T)}$ so there are at least the same cusps on $\mathscr{X}_{\Gamma_0(T)_2}$ and hence there are 2 elliptic points.*

Let $\overline{\Gamma_0(T)_2}$ denote the image of $\Gamma_0(T)_2$ in $\mathrm{GL}_2(A/T) \cong \mathrm{GL}_2(\mathbb{F}_q)$. As in [Gek01, Section 3], let $(A/T)^2_{prim}$ denote the primitive vectors in $A/T \times A/T$, i.e. those vectors which span a non-zero direct summand. Then from [Gek01, Section 3] we know

$$\{\textit{cusps of } X_{\Gamma_0(T)_2}\} \cong \overline{\Gamma_0(T)_2}\backslash(A/T)^2_{prim}/\mathbb{F}_q^\times,$$

so the cusps of $X_{\Gamma_0(T)_2}$ are precisely the $\Gamma_0(T)_2$-orbits of 0 and ∞ which correspond to the primitve vectors $(1,0)$ and $(0,1)$. So, there are exactly these two cusps and no further elliptic points. Let $\alpha \in \mathbb{Q}$ be such that

$$\frac{2k-2l-kq}{k(q-1)} \leqslant \alpha < \frac{2k-2l-kq}{k(q-1)} + 1$$

and the number r of best lower approximations to α with denominator strictly greater than 1 is $r = 2$. Then let

$$\begin{aligned} D \overset{def}{=} K_{\mathscr{X}_{\Gamma_0(T)_2}} + 2\Delta &\sim K_{\mathbb{P}^1} + \alpha(0) + \alpha(\infty) + 2(0+\infty) \\ &= \alpha(\infty) + (\alpha+2)(0), \end{aligned}$$

since $K_{\mathbb{P}^1} = -2\infty$. *We see that*

$$\begin{aligned} h^0\left(\frac{k}{2}D\right) &= 2\left\lfloor\frac{k}{2}(\alpha)\right\rfloor + k + 1 \\ &= k\left(\frac{2k-2l-kq}{k(q-1)}\right) + k + 1 \\ &= 1 + \frac{k-2l}{q-1} \\ &= \dim_C(M_{k,l}(\Gamma_0(T))), \end{aligned}$$

where we know this dimension from [DK23, Proposition 4.1*].*

Finally, we see from [O'D15, Theorem 6*] that the canonical ring* R_D*, i.e. the log stacky canonical ring for* $\mathscr{X}_{\Gamma_0(T)_2}$*, is generated by* 3 *functions,* Δ_T*,* Δ_W *and* E_T *corresponding to* U, V *and* Z *respectively, and has a single relation* $UV - Z^2$*. We include a rough sketch of the monoid* $M \stackrel{def}{=} \{(d, c) \in \mathbb{Z}^2 : -d(\alpha+2) \leqslant c \leqslant d\alpha\}$ *from [O'D15], where generators for* R_D *correspond to shaded-in lattice points in degrees* 2 *and* $q-1$*:*

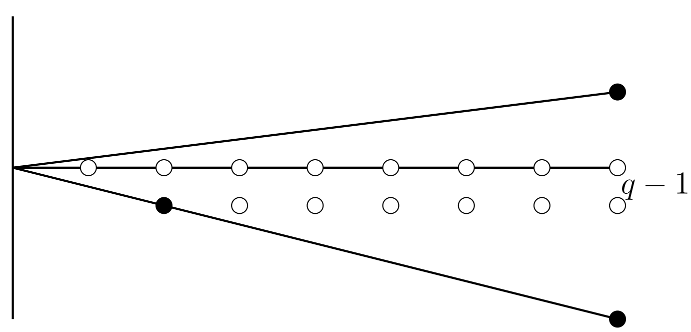


Figure 8.1: Monoid Sketch

8.3 Congruence Subgroups of $\mathrm{SL}_2(A)$ and Other Open Problems

There are two major limiting factors in computing examples of algebras of Drinfeld modular forms for congruence subgroups. The first is that the signature (recall Section 3.3) of many Drinfeld modular curves may not be compatible with existing results for log canonical rings of stacky curves. Worse yet, we may not even be able to get so far, as we need to strongly compare cusps of a congruence subgroup $\Gamma \leqslant \mathrm{GL}_2(A)$ with its square-determinant subgroup $\Gamma_2 \leqslant \Gamma$ to even determine the necessary signature.

Thanks to Mihran Papikian for pointing out the following. The comparison we need between cusps and the genus computation for a Drinfeld modular curve can both be accomplished combinatorially with an examination of the Bruhat-Tits tree $\mathscr{T}$ of $\mathrm{PGL}_2(K_\infty)$. We adopt the philosophy of Serre, described by Gekeler and Nonnengardt [GN95, Introduction], that to study group theoretic properties of a congruence subgroup Γ is similar to considering the action of Γ on $\mathscr{T}$ as a way to comment on what the Bruhat-Tits tree "does" for our algorithm in Section 8.1.

We know from Mumford a genus formula for the modular curves which arise as quotients of $\Omega \cup \mathbb{P}^1(K)$ by a discrete subgroup $\Gamma \leqslant \mathrm{PGL}_2(K_\infty)$ with finite covalence:

$$\begin{aligned}\mathrm{genus}(\Gamma\backslash(\Omega \cup \mathbb{P}^1(K))) &= \mathrm{genus}(\Gamma\backslash\mathscr{T}) \\ &= \dim(H_1(\Gamma\backslash\mathscr{T}, \mathbb{Z})).\end{aligned}$$

This dimension in turn be computed using the theory of [GN95] discussed below. It is well-known that $\mathrm{GL}_2(A)\backslash\mathscr{T}$ is a half-line, and we know from [Ser80] that so is $\mathrm{SL}_2(A)\backslash\mathscr{T}$.

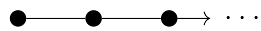


Figure 8.2: The half-line $\mathrm{GL}_2(A)\backslash\mathscr{T}$ *(or* $\mathrm{SL}_2(A)\backslash\mathscr{T}$*)*

The technique of [GN95] is to consider ramified coverings $\pi_\Gamma : \Gamma\backslash\mathscr{T} \to \mathrm{GL}_2(A)\backslash\mathscr{T}$. This allows the authors to derive genus formulae [GN95, Theorem 2.17] for $\Gamma_0(N\backslash\mathscr{T})$, [GN95, Corollary 5.3] for $\Gamma_1(N\backslash\mathscr{T})$ (note the difference in naming convention for subgroups between our Section 4.1 and [GN95, Section 0]) and [GN95, Corollary 5.8] for $\Gamma(N)\backslash\mathscr{T}$.

We end this section by discussing how to approach the comparison of cusps via the Bruhat-Tits tree. Mihran Papikian kindly sketched the following example.

Example 8.3.1. *Let x and $y \in A$ have* $\deg x = \deg y = 1$. *Then one computes from [GN95]*

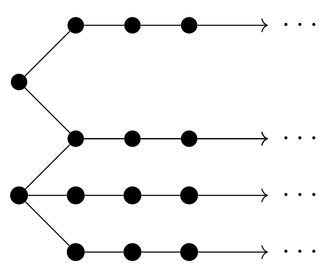


Figure 8.3: [GN95] computes $\Gamma_0(xy)\backslash\mathscr{T}$ *"layer by layer"*

So, by collapsing half-lines into arrows, we have

where in Figure 8.3 half-lines, and in Figure 8.4 arrows, indicate the cusps of $\Gamma_0(xy)$.

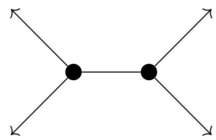


Figure 8.4: $\Gamma_0(xy)\backslash\mathscr{T}$

We can extend the main results of this thesis or [Fra23] by comparing graph quotients $\Gamma_2\backslash\mathscr{T}$ and $\Gamma\backslash\mathscr{T}$ with the constructive algorithm [GN95, (3.8)] to form the necessary comparison between cusps of these congruence subgroups. We hope this enables us to compute further examples of algebras of Drinfeld modular forms for at least the arithmetic subgroups $\Gamma \leqslant \mathrm{GL}_2(A)$. In joint work with Mihran Papikian and Kevin Ho, we hope to extend this theory to graph quotients $\Gamma^1\backslash\mathscr{T}$ for arithmetic congruence subgroups $\Gamma^1 \leqslant \mathrm{SL}_2(A)$. It will be interesting to see whether we will be able to compute algebras of Drinfeld modular forms for congruence subgroups of $\mathrm{SL}_2(A)$ eventually.

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Appendix A

Appendix - Why are Canonical Rings?

In this Appendix we discuss the localization of rings, the infinitesimal lifting criterion and morphisms of schemes. No topic in this chapter is strictly necessary for the proofs of our main results. These are ideas which appear in the literature often enough to merit some treatment here, but the main point we focus on is showing that a curve is isomorphic to its image under a canonical embedding into projective space.

A.1 Localization

Localization is perhaps one of the most important tools for doing calculations in algebraic geometry. The remainder of this chapter relies heavily on the theory we describe in this section. Though arguably elementary, it is nevertheless worth having a notes on this topic on hand at all times. These notes come from [AM21] who provide a more thorough treatment.

Let A be a ring. A **multiplicatively closed subset** is some $S \subseteq A$ such that

$1 \in S$ and $ab \in S$ if a and $b \in S$. If S is some multiplicatively closed subset, we define a relation $\sim$ on $A \times S$ by

$$(a, s) \sim (b, t) \iff (at - bs)v = 0$$

for some $v \in S$. By definition this relation is reflexive and symmetric, so we verify that it is transitive to be sure we have a well-defined equivalence relation.

Proof. Suppose $(a, s) \sim (b, t)$ and $(b, t) \sim (c, u)$. There are some $v, w \in S$ such that $(at - bs)v = 0$ and $(bu - ct)w = 0$, so

$$atvuw - bsvuw = 0 \quad \text{and} \quad buwsv - ctwsv = 0;$$

by adding these together we have $(au - cs)tvw = 0$. Since S is closed under multiplication $tvw \in S$ and we conclude $(a, s) \sim (c, u)$, i.e. $\sim$ is a well-defined equivalence relation. $\square$

Let $a/s \overset{def}{=} [(a, s)] = \{(b, t) \in A \times S : (a, s) \sim (b, t)\}$ denote the equivalence class of (a, s). Let $S^{-1}A$ denote the set of these equivalence classes. Then $S^{-1}A$ is a ring under the operations

$$\begin{cases} a/s + b/t \overset{def}{=} (at + bs)/st \\ a/s(b/t) \overset{def}{=} ab/st. \end{cases}$$

The ring $S^{-1}A$ is a commutative ring with 1, and we have a ring homomorphism $A \to S^{-1}A$ given by $x \mapsto x/1$. We enumerate several facts about this ring.

- For each $s \in S$ we know $s/1 \in (S^{-1}A)^\times$.

- If $a/1 = 0$ then $as = 0$ for some $s \in S$.

- Every element of $S^{-1}A$ has form $(a/1)(s/1)^{-1} = a/s$ for some $a \in A$ and $s \in S$.

- We can determine $S^{-1}A$ up to isomorphism by the former three facts.

We conclude this section with two examples.

Example A.1.1 ([DF04, Example 4; page 708])**.** *Let $V \neq \varnothing$ be some set. Let k be a field. Let R be any ring of k-valued functions on V containing the constant funcions. For any $a \in V$, let M_a denote the ideal of functions in R that vanish at a. Then M_a is the kernel of the evaluate-at-a ring homomorphism $R \to k$ given by $f \mapsto f(a)$. Since R contains the constant functions the evaluation is surjective, so M_a is a maximal ideal. The localization of R at M_a is*

$$R_{M_a} = \{f/g : f, g \in R \textit{ and } g(a) \neq 0\}.$$

Each function in R_{M_a} can be evaluated at a by $(f/g)(a) = f(a)/g(a)$, i.e. R_{M_a} is the ring of k-valued rational functions defined at a.

Example A.1.2 ([Har77, Page 76])**.** *Let S be a graded ring. Let S_+ denote the ideal*

$$S_+ \overset{def}{=} \oplus_{d>0} S_d$$

of S. Let $\mathrm{Proj}(S)$ be the set

$$\mathrm{Proj}(S) = \left\{ p \trianglelefteq S : \begin{array}{c} p \textit{ is a homogeneous prime ideal of } S \\ \textit{which does not contain all of } S_+ \end{array} \right\}.$$

For each $p \in \mathrm{Proj}(S)$, let T be the multiplicative semigroup of homogeneous elements of S which are not in p. Then we have a localized ring $S_p \stackrel{def}{=} T^{-1}S$ and a subring $S_{(p)}$ of degree 0 elements of S_p.

A.2 The Infinitesimal Lifting Property

A strong result in the spirit of claims like "genus $g \geqslant 4$ curves are the intersections of quadrics" or "there are 27 lines on a cubic surface" in algebraic geometry is the fact that "the tangent space of a smooth variety of dimension n has dimension $2n$." In this section we will discuss the infinitesimal lifting property, one of the tools used to prove that fact about tangent spaces.

First, we define a **formal derivation** ∂ as follows:

$$\begin{cases} \partial(x_i^n) \stackrel{def}{=} n x_i^{n-1}\dot{x}_i \\ \partial(cf) \stackrel{def}{=} c\partial(f), \ c \text{ constant} \\ \partial(f+g) \stackrel{def}{=} \partial(f) + \partial(g) \\ \partial(x_i x_j) \stackrel{def}{=} x_i \dot{x}_j + x_j \dot{x}_i. \end{cases}$$

When we equip a polynomial ring with this formal derivation we will be able to describe the tangent bundle of the associated affine variety explicitly. We first define an intermediate object that we use to describe the tangent bundle of a **differential ring**, i.e. a ring with a formal derivation.

Definition A.2.1. *Let $A = R[x_1 \cdots, x_n]/\langle f_1, \cdots, f_e\rangle$. The* ***first jet space*** *A^1 of A*

is the ring

$$A^1 \stackrel{def}{=} A[\dot{x_1}, \cdots, \dot{x_n}]/\langle \partial(f_1), \cdots, \partial(f_e) \rangle,$$

where ∂ is our formal derivation.

Let $A = R[x_1 \cdots, x_n]/\langle f_1, \cdots, f_e \rangle$. Let $Y = \operatorname{Spec} A$. The **tangent bundle** $T_{Y/R}$ on Y is $T_{Y/R} \stackrel{def}{=} \operatorname{Spec}(\operatorname{Sym}(\Omega^1_{Y/R}))$. Now we may reap the benefit of of defining a jet space with the following result.

Lemma A.2.2. *In the terminology above, we have*

$$T_{Y/R} = \operatorname{Spec}(\operatorname{Sym}(\Omega^1_{Y/R})) = \operatorname{Spec}(A^1).$$

Proof. We argue that $\operatorname{Sym}(\Omega^1_{Y/R}) = A^1$. By definition

$$\Omega^1_{Y/R} = \bigoplus_{i=1}^{n} R[x_1, \cdots, x_n]dx_i \cong R[x_1, \cdots, x_n]^{\oplus n},$$

and for V a finite dimensional free R-module with basis $v_1, \cdots, v_m$

$$\operatorname{Sym}(V) \stackrel{def}{=} R[x_1, \cdots, x_m],$$

where x_i is identified with v_i. In other words

$$\operatorname{Sym}(A^{\oplus n}) \cong A[x_1, \cdots, x_n]$$

so

$$\operatorname{Sym}(\Omega^1_{X/R}) \cong R[x_1, \cdots, x_n][dx_1, \cdots, dx_n] \cong A^1.$$

$\square$

Note that in the case of no relations in the definition of A^1, i.e. when $A^1 \stackrel{def}{=} A[\dot{x_1}, \cdots, \dot{x_n}] = R[x_1, \cdots, x_n, \dot{x_1}, \cdots, \dot{x_n}]$, we see that $\dim(A^1) = \dim((\mathbb{A}^n)^2) = 2\dim(\mathbb{A}^n)$.

Now that we understand the tangent bundle to an affine variety, we consider schemes next. There is one trick which is very motivational and useful, but most importantly can be illustrated with a cartoon.

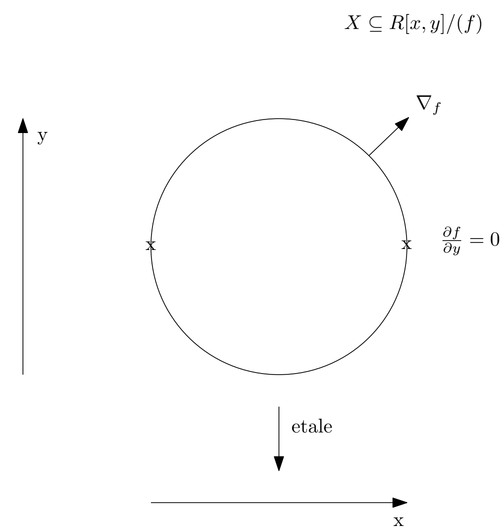

Figure A.1: An Infinitesimals Trick Cartoon

To define an étale morphism from $X \subset \mathbb{A}^2$ to $\mathbb{A}^1$, one of the charts, we need to localize at $\frac{\partial f}{\partial y}$ and use the infinitesimal lifting property. Let $U = D_+(\frac{\partial f}{\partial y})$ be the Zariski open nonvanishing locus of $\frac{\partial f}{\partial y}$. Then $U = \operatorname{Spec}(A_{\frac{\partial f}{\partial y}})$, and we get our étale map by lifting. Recall from considering the relative cotangent sequence (as in [Har77, Page 182]) that if $A \to B$ is an étale morphism of R-algebras, then $\Omega_{B/R} \cong \Omega_{A/R} \otimes_A B$.

Now we define the lifting property which is our main focus.

Definition A.2.3. *A morphism $B \to A$ of R-algebras has the* ***infinitesimal lifting property*** *(resp. is* ***formally smooth****) if and only if for all R-algebras $C \supseteq I$, for I and ideal such that $I^2 = 0$, the following commutes*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, dashed, "\exists"] \\ C \arrow[r] & C/I \end{tikzcd}$$

Figure A.2: Infinitesimal Lifting Property

and there is a lift $A \to C$ of $A \to C/I$, indicated by the dashed arrow.

That is the "what" of the infinitesimal lifting property, but "why" is it so named? The motivation is that we regard elements of I as infinitesimals ε such that $\varepsilon^2 = 0$.

Now our strategy will be to cover a smooth scheme X over a ring R by affine opens $U_i = \mathrm{Spec}(A_i)$ such that there exist étale maps $\varepsilon_i : U_i \to \mathbb{A}_R^{\dim X}$. We begin with an easy exercise.

Lemma A.2.4. $R[x] \to R[x, y]/(f)_{\frac{1}{[\partial f/\partial y]}}$ *has the infinitesimal lifting property.*

Proof. Let $B = R[X]$ and let $A = R[x, y]/(f)_{\frac{1}{[\partial f/\partial y]}}$ and consider the diagram

where $\beta(x) = c$, $\alpha(x) = \bar{c}$ and $\alpha(y) = \overline{d}$ are such that $f(\bar{c}, \overline{d}) = 0$. Then $\tilde{\alpha}(x) = c_1 + c_2$ for some $c_2 \in I$ and $\tilde{\alpha}(y) = d_1 + d_2$ for some $d_2 \in I$ so that $d_2^2 = 0$. Note the same is true for c_2 by definition of I but we can say something even stronger.

$$\begin{tikzcd}
R[x] \arrow[r] \arrow[d, "\beta"] & R[x,y]/(f)_{\frac{1}{[\partial f/\partial y]}} \arrow[d, "\alpha"] \arrow[dl, dashed, "\tilde{\alpha}"] \\
C \arrow[r] & C/I
\end{tikzcd}$$

By definition of β and the commutativity of the diagram, $c_2 = 0$. Therefore we need $f(c, d_1 + d_2) = 0$. Consider the Taylor series

$$f(c, d_1) + \frac{\partial f}{\partial y}(c, d_1) d_2 + O(d_2^2) = 0.$$

Then if $\frac{\partial f}{\partial y}$ is invertible, we can solve for d_2, so consider C/I to be the localization of A at $\frac{\partial f}{\partial y}$, namely $C/I = A_{\frac{\partial f}{\partial y}}$. Then

$$d_2 = \frac{-f}{\frac{\partial f}{\partial y}}(c, d_1),$$

so $\tilde{\alpha}$ is well-defined. $\square$

Now we generalize little-by-little.

Lemma A.2.5. *Suppose $X \subseteq \mathbb{A}^n_R = \operatorname{Spec} R[x_1, \cdots, x_n]/(f)$. Then the maps $R[x_i] \to R[x_1, \cdots, x_n]/(f)_{\frac{1}{[\partial f/\partial x_i]}}$ have the infinitesimal lifting property for all $1 \leqslant i \leqslant n$.*

Proof. This is the same setup as Lemma A.2.4 but with worse notation. Fix some i and consider the diagram

$$\begin{tikzcd}
R[x_i] \arrow[r] \arrow[d, "\beta"] & R[x_1, \cdots, x_n]/(f)_{\frac{1}{[\partial f/\partial x_i]}} \arrow[d, "\alpha"] \arrow[dl, dashed, "\tilde{\alpha}"] \\
C \arrow[r] & C/I
\end{tikzcd}$$

where $\beta(x_i) = c_i$ and $\alpha(x_i) = \overline{c_i}$ for some $\overline{c_i} \in C/I$ such that $f(\overline{c_1}, \cdots, \overline{c_n}) = 0$. As before we have

$$c_i = \beta(x_i) = \tilde{\alpha}(x_i) = c_i + \tilde{c_i}$$

for some $\tilde{c_i} \in I$ so $\tilde{c_i} = 0$. Likewise $\tilde{\alpha}(x_j) = c_j + \tilde{c_j}$ for $\tilde{c_j}^2 = 0$ since it is a member of I, and the vanishing condition $f(\overline{c_1}, \cdots, \overline{c_n}) = 0$ means in the Taylor series we have

$$\begin{aligned} f(c_1 + \tilde{c_1}, \cdots, c_i, \cdots, c_j, \cdots, c_n + \tilde{c_n}) \\ + \frac{\partial f}{\partial x_i}(c_1 + \tilde{c_1}, \cdots, c_i, \cdots, c_{j*}, \cdots, c_n + \tilde{c_n})c_j + O(c_j^2) = 0, \end{aligned} \quad \text{(A.2.1)}$$

where c_{j*} means c_j is removed. As before since we are in the localization at $\frac{\partial f}{\partial x_i}$ we can solve for c_j and $\tilde{\alpha}$ is well-defined. □

We content ourselves with stating one more partial result, as this is sufficient to illustrate how to use this theory while not going overboard with the rather heinous notation.

Lemma A.2.6. *The maps $R[x_i] \to [R[x_0, x_1]/(f_0, f_1)]_{\mathrm{Jac}(f_0;f_1)}$, for $0 \leqslant i, j \leqslant 1$ all have infinitesimal lifting.*

Proof. Now since there are two functions, we need to localize at their Jacobian, denoted $\mathrm{Jac}(f_0; f_1)$. Fix an i either 0 and 1. Consider the diagram

$$\begin{tikzcd} R[x_i] \arrow[r] \arrow[d, "\beta"] & R[x_0, x_1]/(f_0, f_1)_{\frac{1}{[\mathrm{Jac}(f_0;f_1)]}} \arrow[d, "\alpha"] \arrow[dl, dashed, "\tilde{\alpha}"'] \\ C \arrow[r] & C/I \end{tikzcd}$$

where

$$c_i = \beta(x_i) = \tilde{\alpha}(x_i) = c_i + \tilde{c_i},\ \tilde{c_i} \in I \Rightarrow \tilde{c_i} = 0$$

and $\tilde{\alpha}(x_j) = c_j + \tilde{c_j}$ for $j \neq i$, $\tilde{(}c_j) \in I$ so $\tilde{c_j}^2 = 0$ and now we have $f_k(\overline{c_0}, \overline{c_1}) = 0$ for both $k = 0$ and $k = 1$. As always we make use of the localization to compute the infinitesimals $\tilde{c_j}$ for $j \neq i$ (remember i is the coordinate fixed at the beginning). We have

$$\begin{cases} f_0(c_0, c_1) + \det \begin{vmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_1}{\partial x_0} \\ \frac{\partial f_0}{\partial x_1} & \frac{\partial f_1}{\partial x_1} \end{vmatrix} (c_0, c_1)\tilde{c_j} + O(\tilde{c_j}^2) = 0 \\ f_1(c_0, c_1) + \det \begin{vmatrix} \frac{\partial f_0}{\partial x_0} & \frac{\partial f_1}{\partial x_0} \\ \frac{\partial f_0}{\partial x_1} & \frac{\partial f_1}{\partial x_1} \end{vmatrix} (c_0, c_1)\tilde{c_j} + O(\tilde{c_j}^2) = 0 \end{cases}$$

so

$$\tilde{c_j} = \frac{-[f + g]}{2\,\mathrm{Jac}(f_0; f_1)}(c_0, c_1)$$

and $\tilde{\alpha}$ is well-defined. □

A.3 Morphisms of Schemes

We conclude this chapter by discussing how one shows that a curve is isomorphic to its image under a canonical embedding. To start we collect some background material to follow this discussion as it proceeds in [Har77].

Definition A.3.1 ([Har77]). *A* ***morphism of locally ringed spaces*** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a map* $(f, f^\#)$ *with* $f : X \to Y$ *a continuous map of spaces and* $f^\# : \mathcal{O}_Y \to \mathcal{O}_X$ *of sheaves of rings, such that for each point* $P \in X$ *the induced map* $f^\#_P : \mathcal{O}_{Y,f(P)} \to f_*\mathcal{O}_{X,P}$ *on stalks is a local isomorphism of local rings.*

We describe a local homomorphism of local rings as follows. Given a point $P \in X$, the morphism of sheaves $f^{\#}$ from Definition A.3.1 induces ring homomorphisms $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$ for each open $V \subset Y$. As V ranges over all open neighborhoods of $f(P)$, the preimages $f^{-1}(V)$ range over a subset of all neighborhoods of P. We get a map

$$\mathcal{O}_{Y,f(P)} = \varinjlim_{V} \mathcal{O}_Y(V) \to \varinjlim_{V} \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_{X,P}$$

which induces a **local homomorphism** $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$. So, if $m_{f(P)} \trianglelefteq \mathcal{O}_{Y,f(P)}$ and $m_P \trianglelefteq \mathcal{O}_{X,P}$ are the unique maximal ideals of their respective local rings, then $(f_P^{\#})^{-1}(m_P) = m_{f(P)}$.

We can now properly consider how to construct a scheme from a graded ring. Let S be a graded ring. Let S_+ denote the ideal $S_+ \stackrel{def}{=} \oplus_{d>0} S_d$of S. Let $\operatorname{Proj}(S)$ be the set

$$\operatorname{Proj}(S) = \left\{ p \trianglelefteq S : \begin{array}{c} p \text{ is a homogeneous prime ideal of } S \\ \text{which does not contain all of } S_+ \end{array} \right\}.$$

First we describe a topological space. If a is an ideal of S then let $V(a) = \{p \in \operatorname{Proj} S : p \supseteq a\}$. We can define a topology on $\operatorname{Proj}(S)$ by defining $\{\text{closed sets}\} = \{V(a) : a \text{ some homogeneous ideal of } S\}$. Next, we need a sheaf of rings on $\operatorname{Proj}(S)$. Let T be the multiplicative semigroup of homogeneous elements of S which are not in p. Then we have a localized ring $S_p \stackrel{def}{=} T^{-1}S$ and a subring $S_{(p)}$ of degree 0 elements of S_p. For any open subset $U \subseteq \mathrm{Proj}(S)$ define the structure sheaf by

$$\mathcal{O}(U) \stackrel{def}{=} \left\{ s : U \to \bigsqcup_{p \in U} S_{(p)} : \begin{array}{c} \text{for each } p \in U,\ \ s(p) \in S_{(p)} \\ \text{and there is some neighborhood } V \text{ of } p \text{ in } U \\ \text{and some homogeneous } a, f \in S \\ \text{(of the same degree) such that for all } q \in V \\ f \neq q \text{ and } s(q) = a/f \in S_{(q)} \end{array} \right\}.$$

We can now formally define the scheme associated to a graded ring.

Proposition A.3.2 ([Har77, II.2.5])**.** *Let S be a graded ring.*

1. *For any $P \in \mathrm{Proj}(S)$ the stalk $\mathcal{O}_P$ is isomorphic to the local ring $S_{(P)}$.*

2. *For any homogeneous $f \in S_+$, let $D_+(f) \stackrel{def}{=} \{P \in \mathrm{Proj}(S) : f \notin P\}$. Then $D_+(f)$ is an open subset of $\mathrm{Proj}(S)$, these open sets cover $\mathrm{Proj}(S)$, and for each $D_+(f)$ we have isomorphisms*

$$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \mathrm{Spec}(S_{(f)})$$

   as locally ringed spaces, where $S_{(f)} \subset S_f$ is the subring of degree 0 elements.

3. $\mathrm{Proj}(S)$ *is a scheme.*

With the tools we defined in this chapter it is an exercise to argue that if X is any scheme whose canonical bundle K_X is very ample, then as schemes

$$\varphi_{K_X}(X) \cong X \cong \mathrm{Proj}(R(X, K_X)),$$

i.e. a scheme is isomorphic to its image under a canonical embedding, which is Proj of its canonical ring. One need only show that for each point P in the image $\varphi_{K_X}(X)$ the stalk $\mathcal{O}_{\varphi_{K_X}(X),P}$ is isomorphic to a local ring, and that there is a compatible cover of the image $\varphi_{K_X}(X)$ by affine open subschemes, as in Proposition A.3.2.

Example A.3.3. *Suppose X is a genus $g \geqslant 4$ canonical curve over $\mathbb{C}$. We will verify that $\varphi_{\omega_X}(X) \subset \mathbb{P}^{g-1}_{\mathbb{C}}$, the image of X under the canonical embedding, is a well-defined scheme, that it is isomorphic as a scheme to X and that it is isomorphic as a scheme to* $\operatorname{Proj} R(X)$.

First we discuss well-definedness. To show that $\varphi_{\omega_X}(X)$ is scheme we need to show it is a locally ringed space with a cover by open affine subschemes. As a set, $\varphi_{\omega_X}(X) \subset \mathbb{P}^{g-1}_{\mathbb{C}}$ is the collection of points of form $(s_0(P), \cdots, s_{g-1}(P)) \in \operatorname{Proj} \mathbb{C}[x_0, \cdots, x_{g-1}]$, where $P \in X$ is a point. We equip $\varphi_{\omega_X}(X)$ with the induced subspace topology from $\mathbb{P}^{g-1}_{\mathbb{C}}$, where the topology on $\mathbb{P}^{g-1}_{\mathbb{C}}$ is defined by closed sets of the form $V(a) = \{p \in \mathbb{P}^{g-1}_{\mathbb{C}} : p \supseteq a\}$ for a a homogeneous ideal of $\mathbb{C}[x_0, \cdots, x_{g-1}]$.

The more delicate matter is constructing a sheaf of local rings on $\varphi_{\omega_X}(X)$ and we proceed as follows. For each $p \in \varphi_{\omega_X}(X)$ let T be the multiplicative semigroup of homogeneous elements of $\mathbb{C}[x_0, \cdots, x_{g-1}]$ not in p. For this to be well-defined, we need to think of p as a homogeneous prime ideal which does not contain all of $\oplus_{d>0}\mathbb{C}[x_0, \cdots, x_{g-1}]_d$, but since $\varphi_{\omega_X}(X) \subset \operatorname{Proj} \mathbb{C}[x_0, \cdots, x_{g-1}]$ is a subset, $p \in \mathbb{P}^{g-1}_{\mathbb{C}}$ so is indeed such a point. Likewise for each p we define $\mathbb{C}[x_0, \cdots, x_{g-1}]_{(p)} \stackrel{def}{=} (T^{-1}\mathbb{C}[x_0, \cdots, x_{g-1}])_0$ to be the subring of the localized ring $T^{-1}\mathbb{C}[x_0, \cdots, x_{g-1}] = \mathbb{C}[x_0, \cdots, x_{g-1}]_p$ consisting of elements of degree 0. Then for $U \subseteq \varphi_{\omega_X}(X)$ any open *set, we define*

$$\mathcal{O}_{\varphi_{\omega_X}(X)}(U) \overset{def}{=} \left\{ s : U \to \bigsqcup_{p \in \varphi_{\omega_X}(X)} \mathbb{C}[x_0, \cdots, x_{g-1}]_{(p)} \right\}$$

to be the collection of those s such that for $p \in U$, $s(p) \in \mathbb{C}[x_0, \cdots, x_{g-1}]_{(p)}$, and there exists some neighborhood V of p in U and homogeneous $a, f \in \mathbb{C}[x_0, \cdots, x_{g-1}]$ of the same degree such that for all $q \in V$, $f \notin q$ and $s(q) = a/f$ in $\mathbb{C}[x_0, \cdots, x_{g-1}]_{(q)}$.

To verify that $\varphi_{\omega_X}(X)$ is a locally ringed space, we need to show that for each $p \in \varphi_{\omega_X}(X)$, the stalk $\mathcal{O}_{\varphi_{\omega_X}(X),p}$ is isomorphic to some local ring, and in particular the local ring we will use as the target is $\mathbb{C}[x_0, \cdots, x_{g-1}]_{(p)}$. Consider the map $\phi : \mathcal{O}_{\varphi_{\omega_X}(X),p} \to \mathbb{C}[x_0, \cdots, x_{g-1}]_{(p)}$ defined by sending any local section s in a neighborhood of p to its value $s(p)$.

Given any $a/f \in \mathbb{C}[x_0, \cdots, x_{g-1}]_{(p)}$, for a, f homogeneous of the same degree and $f \notin p$, since $D(f)$ is an open neighborhood of p with the section a/f of $\mathcal{O}_{\varphi_{\omega_X}(X)}$ over $D(f)$ whose value at p is a/f, ϕ is surjective.

Given some neighborhood U of p in $\varphi_{\omega_X}(X)$ and $s, t \in \mathcal{O}_{\varphi_{\omega_X}(X)}(U)$ with the same value at p, there is some open neighborhood $V \subseteq U$ containing p such that $s = a/f$ and $b = t/g$ for $a, b, f, g \in \mathbb{C}[x_0, \cdots, x_{g-1}]$ with $\deg a = \deg f$, $\deg b = \deg g$ and $f, g \notin p$. By "the same value at p" we mean that $s \equiv t$ in the local ring, so by definition of the localization $\mathbb{C}[x_0, \cdots, x_{g-1}]_p$ there is some $h \notin p$ such that $h(ga - fb) = 0$ in $\mathbb{C}[x_0, \cdots, x_{g-1}]$ so that $a/f = b/g$ in every local ring $\mathbb{C}[x_0, \cdots, x_{g-1}]_{(q)}$ such that

$f, g, h \notin q$. The set of such q is the intersection $D(f) \cap D(g) \cap D(h)$ which is a finite intersection of open sets containing p, so is an open set containing p. Since $s = t$ in this neighborhood of p, we conclude that s and t have the same stalk at p, so ϕ is injective.

Now we verify that the cover by open affine subschemes of $\mathbb{P}^{g-1}_{\mathbb{C}}$ restricts to such a cover of $\varphi_{\omega_X}(X)$. Once again this is a three part computation where we argue for well-definedness of the restricted cover as a cover, a homeomorphism of spaces from the cover to the embedded curve, and finally an isomorphism of sheaves of local rings.

From the proof of part (2) of Proposition [Har77, II.2.5], we have a cover of $\mathbb{P}^{g-1}_{\mathbb{C}}$ by the open sets indexed by homogeneous $f \in \oplus_{d>0}\mathbb{C}[x_0, \cdots, x_{g-1}]_d$ of form $D_+(f) \overset{def}{=} \{p \in \operatorname{Proj}\mathbb{C}[x_0, \cdots, x_{g-1}] : f \notin p\} = \operatorname{Proj}\mathbb{C}[x_0, \cdots, x_{g-1}] - V(f)$. Further, for each $D_+(f)$ there is an isomorphism of locally ringed spaces

$$(D_+(f), \mathcal{O}_{\mathbb{P}^{g-1}}|_{D_+(f)}) \cong \operatorname{Spec}\mathbb{C}[x_0, \cdots, x_{g-1}]_{(f)}$$

where $\mathbb{C}[x_0, \cdots, x_{g-1}]_{(f)}$ is the subring of $\mathbb{C}[x_0, \cdots, x_{g-1}]_f$ consisting of degree 0 elements in the local ring.

Since X is a canonical curve, φ_{ω_X} is a closed immersion, so by version 2 of [Sta18a, Lemma 29.2.1] for each affine open $D_+(f) \subset \mathbb{P}^{g-1}_{\mathbb{C}}$, there exists some ideal I of $\mathbb{C}[x_0, \cdots, x_{g-1}]_{(f)}$ such that $\varphi^{-1}_{\omega_X}(D_+(f)) \cong \operatorname{Spec}(\mathbb{C}[x_0, \cdots, x_{g-1}]_{(f)}/I)$ as schemes over $D_+(f) \cong \operatorname{Spec}\mathbb{C}[x_0, \cdots, x_{g-1}]_{(f)}$. We finally conclude that $\varphi_{\omega_X}(X)$ is a well-defined scheme.

Now we can work out isomorphisms between the embedded curve, the original curve, and Proj *of the canonical ring. We begin by defining the maps from which we get isomorphisms of schemes* $X \cong \varphi_{\omega_X}(X) \cong \operatorname{Proj} R(X)$. *We have the usual canonical embedding* $\varphi_{\omega_X} : X \to \varphi_{\omega_X}(X)$ *which is base-point free and a closed immersion. For the other isomorphism of schemes, we need a map from* $\mathbb{P}^{g-1}_{\mathbb{C}} \to \operatorname{Proj} R(X)$ *which restricts to an isomorphism on the embedded curve. The isomorphisms*

$$R(X) \otimes_{\mathbb{C}} (\mathbb{C}[x_0, \cdots, x_{g-1}]/I) \cong R[x_0, \cdots, x_{g-1}]/IR[x_0, \cdots, x_{g-1}]$$

given by $r \pmod I \otimes n \mapsto rn \pmod I$ *are well-behaved with respect to localization, so we have a map of affine open subschemes*

$$\varphi^{-1}_{\omega_X} D_+(f) \xrightarrow{\sim} \operatorname{Spec} (R(X)/IR(X))_{(f)},$$

and since those affine opens cover $\varphi_{\omega_X}(X)$ *and* $\operatorname{Proj} R(X)$ *respectively, we get a map of schemes* $\nu : \varphi_{\omega_X}(X) \to \operatorname{Proj} R(X)$ *given by the map on points* $P \mapsto 1 \otimes P$ *corresponding to the base change* $\mathbb{P}^{g-1}_{\mathbb{C}} \to \operatorname{Proj} R(X)$ *restricted to* $\varphi_{\omega_X}(X)$.

Consider the diagrams

$$\begin{tikzcd}
\coprod \varphi^{-1}_{\omega_X}(D_+(f)) \arrow[r] \arrow[d] & \coprod D_+(f) \arrow[r] \arrow[d] & \coprod D_+(a) \arrow[d] \\
X \arrow[r, "\varphi_{\omega_X}"] & \varphi_{\omega_X}(X) \arrow[r, "\nu"] & \operatorname{Proj} R(X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{O}_X \arrow[d] & \mathcal{O}_{\varphi_{\omega_X}(X)} \arrow[l, "\varphi^{\#}_{\omega_X}"'] \arrow[d] & \mathcal{O}_{\operatorname{Proj} R(X)} \arrow[l, "\nu^{\#}"'] \arrow[d] \\
\bigsqcup \mathcal{O}_X|_{\varphi^{-1}_{\omega_X}(D_+(f))} & \bigsqcup \mathcal{O}_{\varphi_{\omega_X}(X)}|_{D_+(f)} \arrow[l] & \bigsqcup \mathcal{O}_{\operatorname{Proj} R(X)}|_{D_+(a)} \arrow[l]
\end{tikzcd}$$

We want to argue that the pairs $(\varphi_{\omega_X}, \varphi^{\#}_{\omega_X})$ *and* $(\nu, \nu^{\#})$ *are isomorphisms of schemes, so the maps on spaces are homeomorphisms and the maps on sheaves of local rings are homomorphisms of sheaves of local rings such that the induced maps on stalks are local isomorphisms of local rings.*

By definition of a closed immersion, X *is homeomorphic to a closed subset in* $\mathbb{P}^{g-1}_{\mathbb{C}}$, *and* $\varphi_{\omega_X}(X)$ *the embedded curve is the only possible choice. From the* $M \otimes_R R/I \cong M/IM$*-style isomorphisms of affine opens in the covers of* $\varphi_{\omega_X}(X)$ *and* $\operatorname{Proj} R(X)$ *we can glue together these homeomorphisms between affine schemes* $D_+(f)|_{\varphi_{\omega_X}(X)}$ *for* $f \in \mathbb{C}[x_0, \cdots, x_{g-1}]$ *homogeneous of strictly positive degree and* $D_+(a)$ *for homogeneous* $a \in R(X)_+$ *into a homeomorphism* ν, *thanks to the scheme structures on the affine opens, i.e. the descent data of the covers.*

Finally we have the induced maps of sheaves of local rings $\varphi^{\#}_{\omega_X} : \mathcal{O}_{\varphi_{\omega_X}(X)} \to (\varphi_{\omega_X})_* \mathcal{O}_X$ *and* $\nu^{\#} : \mathcal{O}_{\operatorname{Proj} R(X)} \to \nu_* \mathcal{O}_{\varphi_{\omega_X}(X)}$ *such that for each point of* X *and* $\varphi_{\omega_X}(X)$ *respectively, the induced maps on stalks are local homomorphisms of local rings. Since* φ_{ω_X} *and* ν *are homeomorphisms, we can strengthen the local homomorphisms on stalks to local isomorphisms since we have set bijections between the affine open covering* *schemes. This means that given* $P \in X$ *and* $\varphi_{\omega_X}(P) \in \varphi_{\omega_X}(X)$ *as* U *and* V *vary over all open neighborhoods of* $\varphi_{\omega_X}(P)$ *and* $\nu(\varphi_{\omega_X}(P))$ *respectively,* $\varphi_{\omega_X}^{-1}(U)$ *and* $\nu^{-1}(V)$ *vary of the entire set of neighborhoods of* P *and* $\varphi_{\omega_X}(P)$ *respectively, so we get maps*

$$\mathcal{O}_{\varphi_{\omega_X}(X),\varphi_{\omega_X}(P)} = \varinjlim_{U} \mathcal{O}_{\varphi_{\omega_X}(X)}(U) \leftrightarrow \varinjlim_{U} \mathcal{O}_X(\varphi_{\omega_X}^{-1}(U)) = \mathcal{O}_{X,P}$$

and

$$\mathcal{O}_{\operatorname{Proj} R(X),\nu(\varphi_{\omega_X}(P))} = \varinjlim_{V} \mathcal{O}_{\operatorname{Proj} R(X)}(V) \leftrightarrow \varinjlim_{V} \mathcal{O}_{\varphi_{\omega_X}(X)}(\nu^{-1}(V)) = \mathcal{O}_{\varphi_{\omega_X}(X),\varphi_{\omega_X}(P)}.$$

This means the local homomorphisms of local rings are local isomorphisms, so we are finished.

Appendix B

Appendix - Syzygies and Cohomology

As in [GL85] and [Gre84] we can describe the homogenous canonical ideal of a curve via calculation of certain Koszul cohomology groups, giving another means to prove Petri's theorem. We develop some tools like the Koszul complex and discuss how Green and Lazarsfeld use them to prove the that the image of a general curve of genus $g \geqslant 4$ under a canonical embedding is cut out by quadrics. The point of doing a purely cohomological version of Petri's theorem, as Green and Lazarsfeld put it, is that when explicitly computing all possible syzygies for an embedded curve as in [Mum99, page 237] or Section 2.5, paring down relations to some minimal set is "unavoidably a bit messy." By using the theory of Green and Lazarsfeld we at least avoid this "messiness."

B.1 Euler Sequence

In this section, some relations between different sheaves are introduced. Eventually, by forming the long exact sequences in sheaf cohomology from different short exact sequences given here, one can use the Koszul cohomology to inductively demonstrate that all syzygies for the canonical ideal of a curve with $g \geqslant 4$ are generated in degree 2.

A fundamental exact sequence we use many times in this section is the following.

Definition B.1.1. *The* ***Euler sequence*** *on $\mathbb{P}^n$ is the following exact sequence of sheaves on $\mathbb{P}^n$*

$$0 \to \Omega_{\mathbb{P}^n_A} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

which relates the sheaf of holomorphic differentials Ω to the structure sheaf on $\mathcal{O}_{\mathbb{P}^n}$.

For rigor, it is worth checking exactness.

Lemma B.1.2 ([Vak02b])**.** *The Euler sequence is exact.*

Proof. Let $\varphi : \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}$ be the degree 1 map

$$(s_0, \cdots, s_n) \mapsto x_0 s_0 + \cdots + x_n s_n.$$

Identifying the kernel of this map with differentials can be done locally since injectivity and surjectivity are local properties. Consider U_0 where $x_0 \neq 0$ some open set. Consider some coordinates $x_{j/0} = \frac{x_j}{x_0}$ for $1 < j \leqslant n$. To each differential

$$f_1(x_{1/0}, x_{2/0}, \cdots, x_{n/0})dx_{1/0} + \cdots + f_n(x_{1/0}, \cdots, x_{n/0})dx_{n/0} \in \Omega_{\mathbb{P}^n}$$

there are $n+1$ sections of $\mathcal{O}(-1)$ since by treating the projective coordinates naively,

$$\begin{aligned} f_1 dx_{1/0} &= f_1 d(\tfrac{x_1}{x_0}) \\ &= f_1 \tfrac{x_0 dx_1 - x_1 dx_0}{x_0^2} \\ &= \tfrac{f_1}{x_0} dx_1 + \tfrac{-x_1}{x_0^2} f_1 dx_0. \end{aligned}$$

Note $x_0 \left(\frac{-x_1}{x_0^2} f_1\right) + x_1 \left(\frac{f_1}{x_0}\right) = 0$ and that both $\frac{-x_1}{x_0^2} f_1$ and $\frac{f_1}{x_0}$ are homogeneous of degree -1.

Let $ß : \Omega_{\mathbb{P}^n_A} \to \mathcal{O}(-1)^{\oplus(n+1)}$ be given by

$$f_1 dx_{1/0} + \cdots + f_n dx_{n/0} \mapsto \left(-\frac{x_1}{x_0^2} f_1 - \cdots - \frac{x_n}{x_0^2} f_n, \frac{f_1}{x_0}, \frac{f_2}{x_0}, \cdots, \frac{f_n}{x_0}\right).$$

First of all $ß|_{U_0}(\Omega_{\mathbb{P}^n}) \subseteq \ker \varphi$ since

$$x_0 \left(-\frac{x_1}{x_0^2} f_1 - \cdots - \frac{x_n}{x_0^2} f_n\right) + x_1 \left(\frac{f_1}{x_0}\right) + \cdots + x_n \left(\frac{f_n}{x_0}\right) = 0$$

Then $ß|_{U_0}$ is one-to-one since $\ker ß|_{U_0} = \{0\}$ as $ß(f_1 dx_{1/0} + \cdots + f_n dx_{n/0}) = (0, \cdots, 0)$ if and only if $f_i = 0$ for $1 \leqslant i \leqslant n$. Also ß is surjective onto the kernel of $\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X$ since for

$$(g_0, \cdots, g_n) \in \ker(\mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O}_X),$$

let $f_i = x_0 g_i$ for each $1 \leqslant i \leqslant n$. To verify this construction consider the map on two different coordinate patches at once, say $U_0 \cap U_1$, where in particular there should be a compatible solution. Note that

$$\begin{aligned}
f_1 dx_{1/0} + \cdots + f_n dx_{n/0} &= f_1 d\tfrac{1}{x_{0/1}} + f_2 d\tfrac{x_{2/1}}{x_{0/1}} + \cdots + f_n d\tfrac{x_{n/1}}{x_{0/1}} \\
&= \tfrac{-f_1}{x_{0/1}^2} dx_{0/1} + \tfrac{x_{0/1}dx_{2/1} - x_{2/1}dx_{0/1}}{x_{0/1}^2} f_2 + \cdots + \tfrac{x_{0/1}dx_{n/1} - x_{n/1}dx_{0/1}}{x_{0/1}^2} f_n \\
&= \tfrac{-f_1}{x_{0/1}^2} dx_{0/1} + \tfrac{f_2}{x_{0/1}} dx_{2/1} - \tfrac{f_2 x_{2/1}}{x_{0/1}^2} dx_{0/1} + \cdots + \tfrac{f_n}{x_{0/1}} dx_{n/1} - \tfrac{f_n x_{n/1}}{x_{0/1}^2} dx_{0/1} \\
&= -\tfrac{f_1 + f_2 x_{2/1} + \cdots + f_n x_{n/1}}{x_{0/1}^2} dx_{0/1} + \tfrac{f_2}{x_{0/1}} dx_{2/1} + \cdots + \tfrac{f_n}{x_{0/1}} dx_{n/1} \\
&= -\tfrac{f_1 + f_2 x_{2/1} + \cdots + f_n x_{n/1}}{x_{0/1}^2} dx_{0/1} + \tfrac{f_2 x_1}{x_0} dx_{2/1} + \cdots + \tfrac{f_n x_1}{x_0} dx_{n/1}.
\end{aligned}$$

In particular the $dx_{2/1}$ term maps to the second factor in $\mathcal{O}(-1)^{\oplus(n+1)}$ and gives $\frac{f_2}{x_0}$ as desired and likewise for each $dx_{j/1}$ term for $j > 2$, indexing factors of $\mathcal{O}(-1)^{\oplus(n+1)}$ from 0 to n. Also, the $dx_{0/1}$ term goes to the "zero factor"

$$\frac{\left(\sum_{j=1}^n f_i \frac{(x_i/x_1)}{(x_0/x_1)^2}\right)}{x_1} = f_1 \frac{x_i}{x_0^2}$$

as desired. The first factor must be corrected because the $\sum_i x_i(i\text{th factor}) = 0$. $\square$

Since the Euler sequence is exact then the following "twist" is exact

$$0 \to \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0, \tag{B.1.1}$$

Let $L = \mathcal{O}_{\mathbb{P}^n}(1) \otimes_{\mathcal{O}_X} \mathcal{O}_X$, let $r = h^0(L) - 1$ and say $M_L = \varphi_L^* \Omega_{\mathbb{P}^r}(1)$. Then the following pullback by φ_L of the sequence B.1.1 above is exact

$$0 \to M_L \to H^0(L) \otimes_{\mathbb{F}} \mathcal{O}_X \to L \to 0. \tag{B.1.2}$$

It is not hard to show that the following is exact

$$0 \to M_L \otimes L^{k-1} \to H^0(L) \otimes_{\mathbb{F}} L^{k-1} \to L \otimes L^{k-1} \to 0.$$

This next Lemma is just another twist, but this time with some wedge products.

Lemma B.1.3. *The following is exact.*

$$0 \to \bigwedge^2 M_L \otimes L^{k-1} \to \bigwedge^2 H^0(L) \otimes_{\mathbb{F}} L^{k-1} \to M_L \otimes L^k \to 0. \tag{B.1.3}$$

Proof. Taking wedge products in B.1.2 and twisting by L^{k-1} also preserves exactness so to obtain the sequence in the statement, first consider the dual sequence

$$0 \to L^\vee \to H^0(L)^\vee \otimes_{\mathbb{F}} \mathcal{O}_X \to M_L^\vee \to 0.$$

By [Sta18b, Tag 00DM] the following is exact

$$L^\vee \otimes H^0(L)^\vee \otimes \mathcal{O}_X \to \bigwedge^2 H^0(L)^\vee \otimes \mathcal{O}_X \to \bigwedge^2 M_L^\vee \to 0.$$

Take the dual again, note that $M_L \cong L \otimes_{\mathcal{O}_X} H^0(L) \otimes_{\mathbb{F}} \mathcal{O}_X$ by B.1.2, and the following is exact

$$0 \to \bigwedge^2 M_L \to \bigwedge^2 H^0(L) \otimes_{\mathbb{F}} \mathcal{O}_X \to M_L$$

and twisting by L^{k-1} finally gives

$$0 \to \bigwedge^2 M_L \otimes L^{k-1} \to \bigwedge^2 H^0(L) \otimes_{\mathbb{F}} L^{k-1} \to M_L \otimes L^k.$$

The rightmost map is given by

$$(s_1 \wedge s_2) \otimes f \mapsto s_1 \otimes s_2 f - s_2 \otimes s_1 f$$

and is surjective since this is a Koszul map $d_{2,k-1}$ (see Definition B.2.2) composed of $(\mathrm{Id} \otimes m_{k-1})$ and $(\psi_{\mathrm{id}} \otimes \mathrm{Id})$, where ψ_{id} is dual to an injective map and is surjective, and m_{k-1} is surjective by definition of the multiplication map in $\bigoplus_{k \in \mathbb{N}} L^k$. This makes the sequence right exact. $\square$

B.2 The Koszul Complex

Before we describe a cohomology that allows us to compute a canonical ideal, we recall the following.

Lemma B.2.1. *Let R be a ring and let M be a free R-module with basis $y_0, \cdots, y_n$. The homogeneous coordinate ring of $\mathbb{P}^n_R$ is $\mathrm{Sym}(M) \cong R[y_0, \cdots, y_n]$.*

Proof. Both the symmetric algebra $\mathrm{Sym}(M)$ and the polynomial ring $R[y_0, \cdots, y_n]$, where the y_i are a basis are free objects in their respective categories. The homogeneous polynomials of degree 1 are a free R-module which can be identified with M itself and in particular satisfies the following universal property of the symmetric algebra: for every linear $f : M \to A$ a morphism of algebras, there is a unique algebra homomorphism $g : \mathrm{Sym}(M) \to A$ such that $f = g \circ i$, for $i : M \to \mathrm{Sym}(M)$ the inclusion map. Suppose that $f' : R[y_0, \cdots, y_n]_1 \to A$ is a linear algebra morphism for some R-algebra A. Then since $R[y_0, \cdots, y_n]$ is the free object in the category of R-algebras there is the unique $g' : R[y_0, \cdots, y_n] \to A$ such that $f' = g' \circ i'$ for

$i' : R[y_0, \cdots, y_n]_1 \hookrightarrow R[y_0, \cdots, y_n].$ $\square$

Now we may define the Koszul cohomology.

Definition B.2.2 ([Gre84, 1.a.2])**.** *Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded* $\mathrm{Sym}(V)$*-module. The* ***Koszul complex*** *is the long exact sequence*

$$\cdots \to \bigwedge^{p+1} V \otimes B_{q-1} \overset{d_{p+1,q-1}}{\to} \bigwedge^{p} V \otimes B_q \overset{d_{p,q}}{\to} \bigwedge^{p-1} V \otimes B_{q+1} \overset{d_{p-1,q+1}}{\to} \bigwedge^{p-2} V \otimes B_{q+2} \overset{d_{p-2,q+2}}{\to} \cdots$$

where the maps $d_{p,q}$ are defined to be the composite maps

$$d_{p,q} = (\mathrm{Id} \otimes m_q) \circ (\Delta' \otimes \mathrm{Id}),$$

$$\text{where } \begin{cases} \Delta' : \bigwedge^p V \to \bigwedge^{p-1} V \otimes V, & \textit{is dual to the exterior product map} \\ m_q : V \otimes B_q \to B_{q+1}, & \textit{is multiplication in } B. \end{cases}$$

The following diagram illustrates the composite boundary maps in the complex from Defintion B.2.2.

$$\begin{tikzcd} \bigwedge^p V \otimes B_q \arrow[r, "\Delta'\otimes\mathrm{Id}"] \arrow[dr, "d_{p,q}"] & \bigwedge^{p-1} V \otimes V \otimes B_q \arrow[d, "\mathrm{Id}\otimes m_q"] \\ & \bigwedge^{p-1} V \otimes B_{q+1} \end{tikzcd}$$

Figure B.1: Maps in the Koszul Complex

We begin by working out each of these maps and verifying that the differentials satisfy $d^2 = 0$ to ensure that we have a well-defined complex. In order to do so we will first introduce a map between wedge products which we can define for any pair of N a free R-module and $\varphi : N \to R$ a morphism.

Consider a diagonalization $\Delta : \bigwedge N \to \bigwedge N \otimes_R \bigwedge N$, the unique map of algebras defined by

$$m \mapsto m \otimes 1 + 1 \otimes m$$

for $m \in \bigwedge^1 N = N$ and $m \otimes 1 + 1 \otimes m \in \bigwedge N \otimes \bigwedge^0 N \oplus \bigwedge^0 N \otimes \bigwedge N \subset \bigwedge N \otimes \bigwedge N$. In particular the component Δ' of Δ which maps $\bigwedge^i N \to N \otimes \bigwedge^{i-1} N$ by

$$\Delta'(m_1 \wedge \cdots \wedge m_i) = \sum_{j=1}^{i} (-1)^{j-1} m_j \otimes m_1 \wedge \cdots \wedge \hat{m_j} \wedge \cdots \wedge m_i,$$

where $\hat{m_j}$ means that m_j is left out of the product, gives a description of the differentials

$$\delta_\varphi : \bigwedge^{i} N \to \bigwedge^{i-1} N$$

in the long exact sequence of wedge products of N. Define δ_φ to be the composite

$$\delta_\varphi : \bigwedge^{i} N \xrightarrow{\Delta'} N \otimes_R \left(\bigwedge^{i-1} N \right) \xrightarrow{\varphi \otimes 1} R \otimes_R \bigwedge^{i-1} N = \bigwedge^{i-1} N.$$

Note when $i = 1$ the composite is just φ.

Let us verify that we have a well-defined complex of wedge products with the differentials δ_φ above.

Lemma B.2.3. *Let $V/\mathbb{F}$ be a finite dimensional vector space over the field $\mathbb{F}$ and* $\mathrm{Sym}(V)$ *be the symmetric algebra over V. Then the differentials $\delta_\varphi : \bigwedge^p V \to \bigwedge^{p-1} V$*

for $\varphi \in V^\vee$ satisfy $\delta_\varphi^2 = 0$.

Proof. Let $\Delta : \bigwedge V \to \bigwedge V$ be the map $x \mapsto x \otimes 1 + 1 \otimes x$. Consider $\Delta' : \bigwedge^p V \to V \otimes \bigwedge^{p-1} V$ given on the basis by

$$\Delta'(m_1 \wedge \cdots \wedge m_p) = \sum_{j=1}^{p} (-1)^{j-1} m_j \otimes m_1 \wedge m_2 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p.$$

Let $\varphi \in V^\vee = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ and let $\delta_\varphi := (\varphi \otimes 1) \circ \Delta'$ be the composite map

$$\bigwedge^p V \xrightarrow{\Delta'} V \otimes \left(\bigwedge^{p-1} V\right) \xrightarrow{\varphi \otimes 1} \bigwedge^{p-1} V.$$

Then

$$\begin{aligned}
\delta_\varphi(m_1 \wedge \cdots \wedge m_p) &= (\varphi \otimes 1)(\textstyle\sum_{j=1}^p (-1)^{j-1} m_j \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p) \\
&= \varphi(\textstyle\sum_{j=1}^p (-1)^{j-1} m_j) \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p \\
&= \textstyle\sum_{j=1}^p (-1)^{j-1} \varphi(m_j) \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p
\end{aligned}$$

and extend by linearity, so

$$\begin{aligned}
\delta_\varphi^2(m_1 \wedge \cdots \wedge m_p) &= \delta_\varphi(\textstyle\sum_{j=1}^p (-1)^{j-1} \varphi(m_j) \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge m_p) \\
&= (\varphi \otimes 1)(\textstyle\sum_{k=1}^p (-1)^{k-1} m_k \otimes \sum_{j=1}^p (-1)^{j-1} \varphi(m_j) \\
&\quad \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge \hat{m}_k \wedge \cdots \wedge m_p) \\
&= \textstyle\sum_{k=1}^p (-1)^{k-1} \varphi(m_k) \otimes \sum_{j=1}^p (-1)^{j-1} \varphi(m_j) \\
&\quad \otimes m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge \hat{m}_k \wedge \cdots \wedge m_p).
\end{aligned}$$

A basis for $\bigwedge^{p-2} V$ as a free $\mathbb{F}$-module of rank $\binom{\dim_{\mathbb{F}}(V)}{p-2}$ is

$$\{v_{i_1}, \cdots, v_{i_{p-2}} : 1 \leqslant i_1 < \cdots < i_{p-2} \leqslant \dim_{\mathbb{F}}(V)\}$$

corresponding to all $(p-2)$-subsets of $\{1, \cdots, \dim_{\mathbb{F}}(V)\}$. So, we can write

$$\delta_\varphi^2(m_1 \wedge \cdots \wedge m_p) = \sum_{l=0}^{\binom{\dim_{\mathbb{F}}(V)}{p-2}} a_l (v_{l_1} \wedge \cdots \wedge v_{l_{p-2}})$$

where the a_l of the term $m_1 \wedge \cdots \wedge \hat{m}_j \wedge \cdots \wedge \hat{m}_k \wedge \cdots \wedge m_p)$ is

$$(-1)^{k-2}(-1)^{j-1}\varphi(m_k)\varphi(m_j) + (-1)^{k-1}(-1)^{j-1}\varphi(m_k)\varphi(m_j) = 0,$$

because for each interchange of m_i and m_j in the wedge to bring m_j out in front, a factor of -1 is added. We conclude that $\delta_\varphi^2 = 0$. □

The next Lemma in this series is a characterization of the map Δ' which appears in the Koszul complex.

Lemma B.2.4. *Let $V/\mathbb{F}$ be a finite dimensional vector space with basis $v_1, \cdots, v_n$. Then the dual to the exterior product map*

$$V^\vee \wedge \bigwedge^{p-1} V^\vee \to \bigwedge^{p} V^\vee$$

given by

$$v^\vee \otimes \alpha \mapsto v^\vee \wedge \alpha$$

is the component Δ' of the diagonal map on the pth graded piece $\bigwedge^P V$ of the exterior *algebra* $\bigwedge V$ *given by*

$$\Delta'(v_1 \wedge \cdots \wedge v_p) = \sum_{j=1}^{p} (-1)^{j-1} v_j \otimes v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_p.$$

Proof. Consider the following diagram

$$\begin{tikzcd}
\alpha \wedge v^\vee & \in & \bigwedge^p V^\vee \arrow[r, leftrightarrow, "\vee"] & \bigwedge^p V \arrow[d] & \ni & v_1 \wedge \cdots \wedge v_p \arrow[d, mapsto] \\
\alpha \otimes v^\vee \arrow[u, mapsto] & \in & \bigwedge^{p-1} V^\vee \otimes V^\vee \arrow[u] \arrow[r, leftrightarrow, "\vee"] & \bigwedge^{p-1} V \otimes V & \ni & \Delta'(v_1 \wedge \cdots \wedge v_p)
\end{tikzcd}$$

Since $(\bigwedge^p V)^\vee = \bigwedge^p V^\vee$, for $v_1^\vee \wedge \cdots \wedge v_p^\vee \in \bigwedge^p V^\vee$ we have $(v_1^\vee \wedge \cdots \wedge v_p^\vee)^\vee = v_1 \wedge \cdots \wedge v_p$, which is abbreviated v. So by definition of Δ'

$$(\Delta'(v))^\vee = \sum_{j=1}^{p} (-1)^{j-1} (v_j \otimes v_1 \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_p)^\vee = \sum_{j=1}^{p} (-1)^{j-1} v_j^\vee \otimes v_1^\vee \wedge \cdots \wedge \hat{v_j} \wedge \cdots \wedge v_p^\vee$$

and applying the exterior product map $v^\vee \otimes \alpha \mapsto \alpha \wedge v^\vee$ to $\Delta'^\vee$ therefore yields

$$\sum_{j=1}^{p} (-1)^{j-1} v_j^\vee \wedge v_1^\vee \wedge \cdots \wedge \widehat{(v_j^\vee)} \wedge \cdots \wedge v_p^\vee = v_1^\vee \wedge \cdots \wedge v_p^\vee.$$

The exterior product map on duals of V is given by

$$(\alpha \otimes v^\vee \mapsto \alpha \wedge v^\vee),$$

which is the dual of Δ'. $\square$

Finally, to conclude our laborious well-definedness checks, it is not hard to check that $d^2 = 0$ in the Koszul complex since we observe that $\delta_\varphi^2 = 0$ from Lemma B.2.3.

Now we may reap the fruits the Koszul cohomology bears. We define the Koszul cohomology groups as follows.

Definition B.2.5 ([Gre84, 1.a.7])**.** *Let $\mathbb{F}$ be a field, let V be an n-dimensional $\mathbb{F}$-vector space and let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded* $\mathrm{Sym}(V)$*-module. The* ***Koszul cohomology groups of*** *B are the groups*

$$K_{p,q}(B,V) = \frac{\ker d_{p,q}}{\operatorname{im} d_{p+1,q-1}}$$

where the maps d come from Definition B.2.2.

Our convention will be that $K_{p,q}(B,V) = 0$ when $p < 0$ or $p > \dim V$.

The reason we have introduced the Koszul cohomology is to give a proof of Petri's theorem. Now we have the tools to consider how Koszul cohomology groups are related to generators and relations for a graded $\mathrm{Sym}(V)$-module. With the appropriate vector space V in mind, it turns out that this calculation will tell us about generators and relations for the canonical ring from Petri's theorem.

To this end, let $B = \bigoplus_{q\in\mathbb{Z}} B_q$ be a graded $\mathrm{Sym}(V)$-module for $V/\mathbb{F}$ some vector space. If $x_1, x_2, \cdots$ are generators for B with $\deg x_i = e_i$ then a **weight q relation among the generators** has form

$$\sum_i u_i x_i, \qquad \text{for some } u_i \in \mathrm{Sym}^{q-e_i}(V).$$

We say such a relation is **primitive** if it is not a $\mathrm{Sym}(V)$-linear combination of relations of lower weight. In the terminology of Section 2.5, *a primitive relation of* *weight q is a (first) syzygy of weight q.* We saw in Section 2.5 that syzygies form groups. These groups have a natural action of the symmetric algebra, and so form a graded $\mathrm{Sym}(V)$-module. We denote the **(pth) syzygies of weight** q by $M_{p,q}(B,V)$ to agree with [Gre84, Definition 1.b.3]. For example:

$M_{0,q}$ is the module of degree q generators for B as a $\mathrm{Sym}(V)$-module;

$M_{1,q}$ is the module of primitive relations in weight q for B, i.e. the (first) syzygies;

$M_{2,q}$ is the module of (first) syzygies of weight q among relations for B, i.e. the (second) syzygies of B;

and so on...

By [Gre84, Theorem 1.b.4], we know

$$K_{p,v}(B,V) \cong M_{p,p+q}(B,V)$$

as $\mathbb{F}$-vector spaces, i.e. the Koszul complex computes syzygies.

B.3 Computing Syzygies with Koszul Cohomology

This section is devoted to the proof of a theorem that Koszul cohomology computes an upper bound for the degree of relations for the ideal of an embedded curve. Let X be a curve over some algebraically closed field $\mathbb{F}$. Suppose L is some very ample line bundle on X with associated embedding $\varphi_L : X \to \mathbb{P}^r$, where $r = h^0(X, L)$. We say that L is **normally generated** if the natural maps $\rho_k : \operatorname{Sym}^k H^0(X, L) \to H^0(X, L^k)$ given by

$$s_1 \otimes s_2 \otimes \cdots \otimes s_k \mapsto s_1 \cdots s_k$$

are surjective for all $k \geqslant 0$. On the other hand, we say some subvariety $V \subset \mathbb{P}^r_{\mathbb{F}}$ is **projectively normal** if the canonical maps $H^0(\mathbb{P}^r_{\mathbb{F}}, \mathcal{O}_{\mathbb{P}^r_{\mathbb{F}}}(d)) \to H^0(V, \mathcal{O}_V(d))$, where $\mathcal{O}_V \cong \mathcal{O}_{\mathbb{P}^r}/I_V$ is the structure sheaf on V, are surjective for all $d > 0$. In fact, *a line bundle L on a curve X is normally generated if and only if the embedded curve $\varphi_L(X) \subset \mathbb{P}^r$ is projectively normal.* Indeed, we have $H^0(X, \varphi_L^* \mathcal{O}_{\mathbb{P}^r_{\mathbb{F}}}(d)) = H^0(X, L^{\otimes d})$, so since $H^0(\mathbb{P}^r_{\mathbb{F}}, \mathcal{O}_{\mathbb{P}^r_{\mathbb{F}}}(d)) = \mathbb{F}[x_0, \ldots, x_r]_d$ and

$$\operatorname{Sym}^k(H^0(X, L)) = \operatorname{Sym}^k(Bs_0 \oplus \cdots \oplus Bs_r) = R[s_0, \ldots, s_r]_k,$$

where $R = R(X, L)$ is the section ring $R(X, L) = \bigoplus_{k \geqslant 0} H^0(X, L^{\otimes k})$, as $\operatorname{Sym}(H^0(X, L))$-modules $\operatorname{Sym}^k(H^0(X, L)) \cong H^0(\mathbb{P}^r_{\mathbb{F}}, \mathcal{O}_{\mathbb{P}^r_{\mathbb{F}}}(k))$ and the equivalence follows.

Recall from the sequence B.1.2 that we define a certain pullback of differentials $M_L = \varphi_L^* \Omega_{\mathbb{P}^r}(1)$. Consider the map $\sigma_k : \bigwedge^2 H^0(X, L) \otimes H^0(X, L^{k-1}) \to H^0(X, M_L \otimes L^k)$ given by

$$(v_1 \wedge v_2) \otimes \alpha \mapsto v_1 \otimes v_2 \alpha - v_2 \otimes v_1 \alpha.$$

Then we can state Green's theorem which bounds the degrees of relations which generate the canonical ideal of X.

Theorem B.3.1 ([GL85, 1.3]). *Suppose L is a normally generated line bundle on a curve X over $\mathbb{F}$ an algebraically closed field. Suppose $k_0 \in \mathbb{Z}$ is such that the maps*

$\sigma_k : \bigwedge^2 H^0(L) \to H^0(M_L \otimes L^k)$ are surjective for all $k \geqslant k_0$. Then every minimal generator for the canonical ideal of X (i.e. every primitive syzygy) in $\mathbb{P}^{g-1}$ has degree at most k_0.

The idea of the proof will be to show commutativity of the diagram below.

$$
\begin{tikzcd}
 & & & \bigwedge^2 H^0(L)\otimes H^0(L^{k-1}) \arrow[dd, "\sigma_k"] \arrow[ddrr] & & & & & & & \\
 & & \bigwedge^2 H^0(L)\otimes \operatorname{Sym}^{k-1} H^0(L) \arrow[ur, "1\otimes\rho_{k-1}"] \arrow[dd, "\beta_k"] \arrow[ddrr] & & & & 0 & & & 0 & \\
 & 0 \arrow[rr] & & \ker \nu_k \arrow[rr] & & H^0(L)\otimes H^0(L^k) \arrow[ur] \arrow[rrr, "\nu_k"] & & & H^0(L^{k+1}) \arrow[ur] \arrow[rr] & & 0 \\
0 \arrow[rr] & & \ker \mu_k \arrow[ur, "\alpha_k"] \arrow[rr] & & H^0(L)\otimes \operatorname{Sym}^k H^0(L) \arrow[ur, "1\otimes\rho_k"] \arrow[rrr, "\mu_k"] & & & \operatorname{Sym}^{k+1} H^0(L) \arrow[ur, "\rho_{k+1}"] \arrow[rr] & & 0 & \\
 & & & H^0(L)\otimes I_k \arrow[ur] \arrow[rrr] & & & I_{k+1} \arrow[ur] & & & & \\
 & & 0 \arrow[ur] & & & 0 \arrow[ur] & & & & &
\end{tikzcd}
$$

Figure B.2: Koszul Cohomology Bounds Syzygies

To show Theorem B.3.1 we rely on another "visual" theorem from topology.

Lemma B.3.2 (snake lemma)**.** *If the following commutes*

$$
\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}
$$

the sequence

$$\ker(a) \to \ker(b) \to \ker(c) \xrightarrow{d} \operatorname{coker}(a) \to \operatorname{coker}(b) \to \operatorname{coker}(c)$$

is exact, where d denotes a connecting homomorphism.

Proof. See [DF04, page 792]. $\square$

We also use the following Lemma to prove Theorem B.3.1.

Lemma B.3.3 (Symmetric-Tensor-Exterior algebra sequence)**.** *Let M be a free R-module of rank n, where R contains $\frac{1}{2}$. Then the following sequence is exact*

$$0 \to \operatorname{Sym}^2 M \to M \otimes M \to \bigwedge^2 M \to 0.$$

Proof. Let $\wedge : M^{\otimes 2} \to \bigwedge^2 M$ be the map $a \otimes b \mapsto a \wedge b$. To make this explicit, suppose $a = \sum_{i=1}^n a_i x_i$ and $b = \sum_{j=1} b_j x_j$. Then the exterior algebra relation $x \otimes x = 0$ forces $(x+y) \otimes (y+x) = 0$ and $(x \otimes y) + (y \otimes x) = 0$, which means

$$\begin{aligned} a \wedge b &= (\textstyle\sum_{i=1}^n a_i x_i) \wedge (\sum_{j=1}^n b_j x_j) \\ &= a \otimes b - \textstyle\sum_{i=1}^n a_i x_i \otimes b_i x_i. \end{aligned}$$

The exterior algebra $\bigwedge M$ is a well known quotient of $\bigoplus_{n \in \mathbb{N}} M^{\otimes n}$ and the map $\wedge$ is surjective. Let $s : \operatorname{Sym}^2 M \to M^{\otimes 2}$ be the map $m_1 m_2 \mapsto \frac{1}{2} \sum_{\sigma \in S_2} m_{\sigma(1)} \otimes m_{\sigma(2)} = \frac{1}{2} \left[m_1 \otimes m_2 + m_2 \otimes m_1 \right]$.
Since $ab - (-1)^{\deg a \deg b} ba = 0$ in $\operatorname{Sym}^2(M)$,

$$\begin{aligned} m_1 \otimes m_2 + m_2 \otimes m_1 &= 0 \\ \iff \quad m_1 m_2 &= (-1)^{\deg a \deg b} m_2 m_1 \\ \iff &\begin{cases} m_1 m_2 = 0, & \text{or} \\ m_1 = m_2 & \text{with } \deg m_1 \text{ odd,} \end{cases} \end{aligned}$$

and in the latter case $m_1^2 = 0 \in \operatorname{Sym}^2 M$. So $\ker(s) = 0$ and s is injective. Then $\operatorname{im}(s) \subset \ker(\wedge)$ since

$$m_1 \wedge m_2 + m_2 \wedge m_1 = m_1 \wedge m_2 - m_1 \wedge m_2 = 0$$

and finally $\ker(\wedge) \subset \operatorname{im}(s)$ since

$$(x \otimes y) + (y \otimes x) = 2s(xy)$$

and

$$(x+y)\otimes(y+x) = s((x+y)(y+x)) - s((y+x)(x+y)).$$

$\square$

Proof of Theorem B.3.1. Every minimal generator for the canonical ideal of X in $\mathbb{P}^r$ has degree at most k_0 if and only if the maps $H^0(L)\otimes I_k \to I_{k+1}$ are surjective for all $k \geqslant k_0$. This statement means that I_{k_0} generates I as a graded ring. Let $\rho_k : \operatorname{Sym}^k H^0(L) \to H^0(L^k)$ be the surjective maps from the definition of a normally generated line bundle. Then $\ker \rho_k = I_k$ and the following commutes.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker(\mu_k) \arrow[r, "\alpha_k"] \arrow[d] & \ker(\nu_k) \arrow[d] & \\
0 \arrow[r] & H^0(L)\otimes I_k \arrow[r] \arrow[d] & H^0(L)\otimes \operatorname{Sym}^k(H^0(L)) \arrow[r, "1\otimes\rho_k"] \arrow[d, "\mu_k"] & H^0(L)\otimes H^0(L^k) \arrow[r] \arrow[d, "\nu_k"] & 0 \\
0 \arrow[r] & I_{k+1} \arrow[r] & \operatorname{Sym}^{k+1}(H^0(L)) \arrow[r, "\rho_{k+1}"] \arrow[d] & H^0(L^{k+1}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where the vertical maps are multiplication in their respective graded rings. The lower horiztonal short exact sequence is the exact sequence induced by the assumption that ρ_{k+1} is surjective and the likewise the upper short exact sequence is induced by ρ_k but with the tensor preserving exactness. It is nontrival that the tensor preserves exactness but this follows from right exactness of the ρ-sequences per [Sta18d, Tag 00CW]. By the snake lemma B.3.2, $H^0(L)\otimes I_k \to I_{k+1}$ is surjective if $\alpha_k : \ker \mu_k \to \ker \nu_k$ is surjective. There are two Koszul complexes with maps that takes values in $\ker \mu_k$

and $\ker \nu_k$ respectively and the normal generation of the line bundle relates these complexes so it is possible to show that α_k is surjective with a computation in Koszul cohomology.

Let $\beta_k = d_{2,k-1}^{(\mathrm{Sym}\, H^0(L))} : \bigwedge^2 H^0(L) \otimes \mathrm{Sym}^{k-1} H^0(L) \to H^0(L) \otimes \mathrm{Sym}^k H^0(L)$ be the maps

$$(v_1 \wedge v_2) \otimes \alpha \mapsto v_1 \otimes (v_2 \cdot \alpha) - v_2 \otimes (v_1 \cdot \alpha).$$

Then β_k is realized in $\ker \mu_k$ as the symmetric relation $x \otimes y - y \otimes x = 0$ forces

$$v_1 \otimes (v_2 \cdot \alpha) - v_2 \otimes (v_1 \cdot \alpha) \mapsto \alpha \otimes (v_1 \otimes v_2) - \alpha \otimes (v_2 \otimes v_1) = 0.$$

By definition B.2.2 $\beta_k = (\mathrm{Id}_{\mathrm{H^0(L)}} \otimes \mu_{k-1}) \circ (\Delta' \otimes \mathrm{Id}_{\mathrm{Sym^{k\text{-}1}\, H^0(L)}})$, ψ_{id} is dual to the exterior product which is injective, and μ_{k-1} is surjective so β_k is surjective onto $\ker \mu_k$.

Turning to $\ker \nu_k$, recall the pullback of the Euler sequence on $\mathbb{P}^r$ from B.1.2. Twist the sequence by L^k and take global sections so that the following is exact

$$H^0(M_L \otimes L^k) \xrightarrow{f} H^0(L) \otimes H^0(L^k) \xrightarrow{\nu_k} H^0(L^{k+1})$$

and $\ker \nu_k = f_* H^0(M_L \otimes L^k)$. To make it more clear how Koszul cohomology will compute these global sections, the pushfoward will be abusively written as just $H^0(M_L \otimes L^k)$, but keep in mind that this is $H^0(M_L \otimes L^k) \subset H^0(L) \otimes H^0(L^k)$. The global sections of the sequence from Lemma B.1.3 form the exact sequence

$$\bigwedge^2 H^0(M_L) \otimes H^0(L^{k-1}) \to \bigwedge^2 H^0(L) \otimes H^0(L^k) \xrightarrow{\sigma_k} H^0(M_L \otimes L^k)$$

where $\sigma_k = d_{2,k-1}^{(\bigoplus_{k\in\mathbb{N}} H^0(L^k))}$ is a Koszul map with exact the same form at β_k but the Koszul complex is with respect to a different graded algebra over $H^0(L)$. Just as with β_k, $\operatorname{im}\sigma_k \subseteq \ker\nu_k$ but this time the matter is subtler, since there is apparently no symmetric relation to fall back on. But

$$\begin{aligned} \nu_k(\sigma((s_1 \wedge s_2) \otimes f)) &= \nu_k(s_1 \otimes s_2 f - s_2 \otimes s_1 f) \\ &= s_1 s_2 f - s_2 s_1 f \\ &= 0, \end{aligned}$$

since ρ_{k-1} is surjective so s_1 and s_2 are the image of some symmetric tensors and $s_1 s_2 = s_2 s_1$. Then the following commutes

$$\begin{tikzcd} \bigwedge^2 H^0(L) \otimes \operatorname{Sym}^{k-1} H^0(L) \arrow[r, "\beta_k"] \arrow[d, "1\otimes\rho_{k-1}"] & \ker\mu_k \arrow[d, "\alpha_k"] \\ \bigwedge^2 H^0(L) \otimes H^0(L^{k-1}) \arrow[r, "\sigma_k"] & \ker\nu_k \end{tikzcd}$$

and if σ_k is surjective then so is α_k. $\square$

B.4 Noether's theorem

In this section, Noether's theorem is proved as a consequence of Theorem B.3.1. This is another tool we use in Petri's theorem.

Theorem B.4.1 ([GL85, Noether]). *A canonically embedded nonhyperelliptic curve $X \subseteq \mathbb{P}^{g-1}$ with genus g is projectively normal. That is to say the maps*

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(k)) \to H^0(X, \Omega_X^k)$$

are surjective for all $k \geqslant 0$.

One useful fact to have on hand for the proof of Noether's theorem is the example of the wedged pulled back Euler sequence B.1.3. Let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^r}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual. The following is exact.

$$0 \to Q_\Omega \otimes \Omega^{-l-1} \to \left(\bigwedge^2 H^0(\Omega)^\vee \right) \otimes_{\mathbb{F}} \Omega^{-l} \to \left(\bigwedge^2 Q_\Omega \right) \otimes \Omega^{-l} \to 0. \tag{B.4.1}$$

This next Lemma introduces an exact sequence which is derived under the assumption that the line bundle being studied is very ample.

Lemma B.4.2 ([GL85, page 7])**.** *Let X be a non-hyperelliptic genus $g \geqslant 4$ canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^{g-1}$ be the map obtained from global sections of the canonical bundle. Let $D = x_1 + \cdots + x_{g-2} \in \mathrm{Div}(X)$, where the x_i are points in X of general position which are distinct and linearly independent in $\mathbb{P}^{g-1}$. Let Λ_D be the $(g-3)$-plane in $\mathbb{P}^{g-1}$ spanned by the points supporting D. Let $L = \Omega(-D)$ and suppose that L is very ample. Then*

1. *Λ_D is the subspace $\mathbb{P}(W_D) \subset \mathbb{P}(H^0(\Omega))$ where $W_D = H^0(\Omega)/H^0(\Omega(-D))$.*
2. *There is a surjection of sheaves on X, $u_D : W_D \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega \otimes \mathcal{O}_D$.*
3. *$\Lambda_D \cap X = D$ as schemes.*
4. *$h^0(\Omega(-D)) = 2$.*
5. *$M_{\Omega(-D)} = \Omega^\vee(D)$.*
6. *Let $\Sigma_D = \ker u_D$. Then $\Sigma_D \cong \oplus_{i=1}^{g-2} \mathcal{O}_X(-x_i)$.*

Proof.

1. The line bundle $L = \Omega(-D)$ is very ample if the induced map φ_L is a closed immersion. In other words L separates points and tangent vectors and hence there is a hyperplane, some global section $s_i \in H^0(X, L)$, which passes through each x_i and not the others. So, the image of D under the immersion are those global sections of $H^0(\Omega)$ which correspond to hyperplanes intersecting in exactly $x_1, \cdots, x_{g-2}$, i.e. the set

$$W_D = \{s \in H^0(\Omega) : \operatorname{div} s + D = 0\} = H^0(\Omega)/H^0(\Omega(-D)).$$

2. Recall from B.1.2 the sequence

$$0 \to M_\Omega \to H^0(\Omega) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega \to 0.$$

The map u_D corresponds to the map $H^0(\Omega) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega$ given by the pullback by φ_L of

$$(s_0, \cdots, s_{g-2}) \mapsto x_0 s_0 + \cdots + x_{g-2} s_{g-2}$$

and therefore is given by a map of the same form. The correspondence is in the sense of the diagram [GL85, 2.1] abbreviated below u_D is surjective since the

$$\begin{array}{ccc} H^0(\Omega) \otimes_{\mathbb{C}} \mathcal{O}_X & \longrightarrow & \Omega \\ \downarrow & & \downarrow \\ W_D & \xrightarrow{u_D} & \Omega \otimes \mathcal{O}_D \end{array}$$

Euler sequence is exact.

3. D is naturally a subscheme of $\mathbb{P}(W_D)$ so since by assumption D spans Λ_D this step follows from the dinstinctness and independence of points in general

position.

4. By Riemann-Roch, since $\deg D = (g-3) < 2g-1$,

$$\begin{aligned} h^0(X, L) - h^0(X, K_X \otimes L^{-1}) &= \deg L + 1 - g \\ h^0(X, L) - (2g-2-2g-2) &= g - 3 + 1 - g \\ h^0(X, L) - 4 &= -2 \end{aligned}$$

so $h^0(\Omega(-D)) = 2$.

5. Recall that $M_{\Omega(-D)}$ is defined by $\varphi^*_{\Omega(-D)}\Omega_{\mathbb{P}^{g-1}}(1)$. To identify this with $\Omega^\vee(D) = T_X(D)$ consider another version of B.1.2

$$0 \to M_{\Omega(-D)} \to H^0(\Omega(-D)) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega(-D) \to 0$$

which is exact since $L = \Omega(-D)$ is very ample by assumption. The original version of the Euler sequence Definition B.1.1 twisted by D is the exact sequence

$$0 \to \Omega_{\mathbb{P}^{g-1}}(D) \to \mathcal{O}_{\mathbb{P}^{g-1}}(D-1)^{\oplus g} \to \mathcal{O}_{\mathbb{P}^{g-1}}(D) \to 0$$

so since by the previous part of the lemma $h^0(\Omega(-D)) = 2$, taking the $\mathcal{O}_X$-duals the pullbacks of the Euler sequences must give the same exact sequences.

6. This is a decomposition of the maps with form $(s_0, \cdots, s_{g-2}) \to x_0 s_0 + \cdots + x_{g-2}s_{g-2}$ into the components $s_i \mapsto x_i s_i$.

$\square$

Next we consider another sequence of vector bundles.

Lemma B.4.3 ([GL85, 2.3])**.** *Let X be a non-hyperelliptic genus $g \geqslant 4$ canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^{g-1}$ be the map obtained from global sections of the canonical bundle. Write $\Omega = \omega_X$, let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^{g-1}}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual.*

1. *The following is exact*

$$0 \to M_{\Omega(-D)} \to M_\Omega \to \Sigma_D \to 0,$$

2. *By Lemma B.4.2 the following is exact*

$$0 \to \Omega^\vee(D) \to M_\Omega \to \oplus_{i=1}^{g-2} \mathcal{O}_X(-x_i) \to 0.$$

Proof.

1. Recall the definitions $M_{\Omega(-D)} = \varphi_{\Omega(-D)}^* \Omega_{\mathbb{P}^{g-1}}(1)$ and $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^{g-1}}(1)$. Let $i : D \hookrightarrow X$ be the inclusion of the divisor. Since D is effective and very ample by assumption and X is projective the map is a closed immersion so there is an exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_* \mathcal{O}_D \to 0$$

where the maps are respectively the inclusion of regular functions which vanish at $-D$ and the quotient map by that inclusion. Taking Euler sequences (vertically, on each term) gives the following exact sequence

$$0 \to \Omega_X(-D) \to \Omega_X \to \Omega_X \otimes \mathcal{O}_D \to 0.$$

This is just inclusion of holomorphic differentials with fixed zeros followed by the quotient by the inclusion. The pullbacks need to commute with this sequence which makes

$$0 \to M_{\Omega(-D)} \to M_{\Omega} \to \Sigma_D \to 0$$

exact, so now the inclusion is happening on the curve itself rather than in the projective space containing the embeddings.

2. In this proof let $g = 5$ for ease of notation, so that $D = x_1 + x_2 + x_3$ and Λ_D is the 2-plane in $\mathbb{P}^4$ spanned by these points. Consider the flag of linear spaces $\Lambda_0 \subset \Lambda_1 \subset \Lambda_D$ corresponding to the divisors $D_0 = x_1$, $D_1 = x_1 + x_2$ and D itself respectively. Let $E_0 = D_0 = x_1$, let $E_1 = x_2$ and let $E_2 = x_3$. Then there is filtration of Σ_D by vector bundles

$$\Sigma_D \supset F_1 \supset F_2 \supset 0$$

such that $F_i/F_{i+1} = \mathcal{O}_X(-E_i)$ by [Sta18e, Tag 0120].

□

This next result about global sections allows for a proof of a 'dual version' of Noether's theorem.

Lemma B.4.4 ([GL85, Corollary 2.4])**.**
Let X be a non-hyperelliptic genus $g \geqslant 4$ canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^{g-1}$ be the map obtained from global sections of the canonical bundle. Write $\Omega = \omega_X$, let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^{g-1}}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the

$\mathcal{O}_X$*-dual. Then for each* $l \geqslant 1$,

$$H^0(Q_\Omega \otimes \Omega^{-l}) = 0 \quad \textit{and} \quad H^0\left(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}\right) = 0.$$

Proof. Consider $Q_\Omega = M_\Omega^\vee$, where

$$M_\Omega = \phi_L^* \Omega_{\mathbb{P}^{g-1}}(1) = \phi_L^* \Omega_{\mathbb{P}^{g-1}} \otimes L.$$

Taking the dual of the exact sequence Lemma B.4.3, and then tensoring by Ω^{-l} gives the sequence

$$0 \to (\oplus_{i=1}^{g-2} \mathcal{O}_X(x_i)) \otimes \Omega^{-l} \to Q_\Omega \otimes \Omega^{-l} \to \Omega(-D) \otimes \Omega^{-l} \to 0.$$

The induced long exact sequence is

$$H^0((\oplus_{i=1}^{g-2} \mathcal{O}_X(x_i)) \otimes \Omega^{-l}) \to H^0(Q_\Omega \otimes \Omega^{-l}) \to H^0(\Omega_X(-D) \otimes \Omega^{-l}) \to \cdots$$

where

$$H^0((\oplus_{i=1}^{g-2} \mathcal{O}_X(x_i)) \otimes \Omega^{-l}) = H^0(\oplus_{i=1}^{g-2} \Omega^{-l}(x_i)) = \oplus_{i=1}^{g-2} H^0(-lK_X + x_i),$$

for K_X a canonical divisor, and where

$$H^0(\Omega(-D) \otimes \Omega^{-l}) = H^0(K_X - D - lK_X) = H^0(-(l-1)K_X - D).$$

Since $\deg(-lK_X + x_i) < 0$, $H^0(-lK_X + x_i) = 0$ and likewise since $\deg -(l-1)K_X -$

$D) < 0$ for all $l \geqslant 1$, both of the H^0's surrounding $H^0(Q_\Omega \otimes \Omega^{-l})$ are 0 and $H^0(Q_\Omega \otimes \Omega^{-l}) = 0$. Taking the induced long exact sequence from (B.4.1),

$$H^0(Q_\Omega \otimes \Omega^{-l-1}) \to \bigwedge^2 H^0(\Omega)^\vee \otimes H^0(\Omega^{-l}) \to H^0\left(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}\right) \to \cdots,$$

by the argument above

$$\bigwedge^2 H^0(\Omega)^\vee \otimes H^0(\Omega^{-l}) \cong H^0\left(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}\right)$$

and again by the argument above the right hand side vanishes by degree considerations. $\square$

This next lemma is equivalent to Theorem B.3.1 if Lemma B.4.4 and Noether's theorem hold. It is a purely cohomological version of Petri's theorem.

Lemma B.4.5 ([GL85, Corollary 1.7])**.** *Let X be a non-hyperelliptic genus $g \geqslant 4$ canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^{g-1}$ be the map obtained from global sections of the canonical bundle. Write $\Omega = \omega_X$, let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^{g-1}}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual. Suppose $H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$ for all $l \geqslant 1$ and the map*

$$\bigwedge^2 H^0(\Omega)^\vee \to H^0\left(\bigwedge^2 Q_\Omega\right)$$

from the sequence (B.4.1) *is surjective. Then the homogeneous ideal of X in its canonical embedding is generated by quadrics.*

Proof. By Lemma B.4.4 $H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$ and the map $\bigwedge^2 H^0(\Omega)^\vee \to H^0(\bigwedge^2 Q_\Omega)$ is injective. Therefore it is enough to show that $\dim H^0(\bigwedge^2 Q_\Omega) = \dim \bigwedge^2 H^0(\Omega)^\vee =$

$\binom{g}{2}$ to conclude that the map in the statement is surjective. By Noether's theorem, Ω is normally generated since it is projectively normal in its embedding and nonhyperelliptic, so the maps ρ_k from Theorem B.3.1 are surjective for $k \geqslant 0$. The punchline of this lemma is a specific version of Theorem B.3.1 so the game is to show the maps σ_k from $B.3.1$ are surjective for $k \geqslant 2$. Let $l = k - 2$ and let ψ_k be the maps in the long exact sequence induced by the sequence (B.4.1)

$$H^0(Q_\Omega \otimes \Omega^{-l-1}) \to \bigwedge^2 H^0(\Omega)^\vee \otimes H^0(\Omega^{-l}) \xrightarrow{\psi_{l+2}} H^0\left(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}\right) \to \cdots.$$

Note that ψ_k is surjective for $k \geqslant 2$ by the hypotheses, but in practice the important feature of these maps is their transpose. Recall the sequence Lemma B.1.3 where wedge products of a pullback of Euler are twisted by L^{k-1}, and write down the long exact sequence

$$\cdots \to H^0(M_L \otimes L^k) \to H^1\left(\bigwedge^2 M_L \otimes L^{k-1}\right) \xrightarrow{\tau_k} \bigwedge^2 H^0(L) \otimes H^1(\Omega^{k-1}) \to \cdots.$$

By duality τ_k is the transpose ψ_k^T, so since ψ_k is surjective, τ_k is injective, but τ_k is injective if and only if σ_k is surjective. By Theorem B.3.1 the homogeneous ideal of X in its embedding is generated by quadrics. $H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$, □

Finally, with the tools used to prove Lemma B.4.5 in mind, Noether's theorem can be proved.

Proof of Theorem B.4.1. Recall that Ω is normally generated if and only if the maps $H^1(M_\Omega \otimes \Omega^{-k}) \to H^0(\Omega) \otimes H^1(\Omega^k)$, from the twist of the pulled-back Euler sequence B.1.2, are injective by Lemma B.4.4. But given Lemma B.4.5, those maps are injective if and only if the injective maps $H^0(\Omega)^\vee \to H^0(Q_\Omega)$ are surjective. Let $M_{\Omega(-D)} = \Omega^\vee(D)$ Recall that Ω is very ample by assumption, and the following sequence, a filtration of M_Ω, is exact

$$0 \to M_{\Omega(-D)} \to M_\Omega \to \oplus_{i=1}^{g-2} \mathcal{O}_X(-x_i) \to 0.$$

Therefore

$$\begin{aligned} h^0(Q_\Omega) &\leqslant h^0(\Omega(-D)) && + \textstyle\sum_{i=1}^{g-2} h^0(\mathcal{O}_X(x_i)) \\ &= 2 && +(g-2) \\ &= h^0(\Omega). \end{aligned}$$

□

B.5 Koszul Cohomological Proof of Petri's Theorem

In this section, the Koszul cohomology will be used to prove Petri's theorem for the case of a genus $g \geqslant 4$, non-exceptional curve. Let X be a non-hyperelliptic, smooth, irreducible, projective complex curve of genus $g \geqslant 4$. To prove Petri's result that $I_{X/\mathbb{P}^{g-1}}$ is generated by quadrics, [GL85] use Lemma B.4.5. The essence of the argument is to demonstrate that

$$h^0\left(\bigwedge^2 Q_\Omega\right) \leqslant \binom{g}{2}$$

since this condition is sufficient for the Lemma B.4.5 to apply by [GL85, Remark 1.9].

We know from Lemma B.4.4 that $H^0(\bigwedge^2 Q_\Omega \otimes \Omega^{-l}) = 0$ for all $l \geqslant 1$, and that the map $\bigwedge^2 H^0(\Omega)^\vee \to H^0(\bigwedge^2 Q_\Omega)$ induced from Lemma B.4.1 is injective. We will use the following version of the uniform position theorem of [ACG11, page 112].

Lemma B.5.1 ([GL85, page 10])**.** *An effective divisor E of degree k spans a $(k-r-1)$-plane in $\mathbb{P}^{g-1}$ if and only if it moves in a linear system of dimension r.*

Let X be a non-hyperelliptic, smooth, irreducible, projective, complex curve of genus $g \geqslant 4$. Let $A \in W_4^1(X)$ be a degree 4 line bundle on X with $h^0(X, A) = 2$ such that A and $\omega_X \otimes A^\vee$ are generated by global sections. Let $D = (\operatorname{div} f)$ for some $f \in H^0(X, A)$. Since A is generated by global sections and all of the spaces in consideration lie over $\mathbb{C}$ which has characteristic 0

$$D = x_1 + \cdots + x_{g-1},$$

for some distinct x_i. Then no effective divisor contained in D can move in a nontrivial linear series. Indeed, suppose such a divisor existed. Then $|D|$ either has a base-point or dimension at least 2, both of which contradict global generation and uniform position per Lemma B.5.1. In $\mathbb{P}^{g-1} = \operatorname{Proj}(H^0(X, \omega))$ this means D spans a $(g-3)$-plane Λ_D, and by Lemma B.5.1 any proper subset of the x_i are linearly independent.

Let $\varphi : X \to \mathbb{P}^{g-1}$ be the map obtained from global sections of the canonical bundle. Let $D = x_1 + \cdots + x_{g-1}$ for some distinct closed points x_i in general position. Write $\Omega = \omega_X$, let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^{g-1}}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual. Let

$M_{\Omega(-D)} = \Omega^\vee(D)$. Finally, let Σ_D be as in Lemma B.4.2. Then it is not hard to show that the following is exact

$$0 \to M_{\Omega(-D)} \to M_\Omega \to \Sigma_D \to 0. \tag{B.5.1}$$

We will use one last Lemma to prove Petri's theorem.

Lemma B.5.2 ([GL85, 3.2])**.** *Let X be a non-hyperelliptic genus $g \geqslant 4$ canonical, smooth, irreducible, complex algebraic curve and let $\varphi : X \to \mathbb{P}^{g-1}$ be the map obtained from global sections of the canonical bundle. Let $D = x_1 + \cdots + x_{g-1}$ for some distinct closed points x_i in general position. Write $\Omega = \omega_X$, let $M_\Omega = \varphi_\Omega^* \Omega_{\mathbb{P}^{g-1}}(1)$ and let $Q_\Omega = M_\Omega^\vee$ be the $\mathcal{O}_X$-dual. Let $\Sigma_D = \ker u_D$ be as in Lemma B.4.2. Then the sequence*

$$0 \to \mathcal{O}_X(-x_{g-2} - x_{g-1}) \to \Sigma_D \to \oplus_{i=1}^{g-3} \mathcal{O}_X(-x_i) \to 0$$

is exact.

Proof. Let $D' = x_1 + x_2$ and let $E = x_{g-2} + x_{g-1}$. Then $\Omega(-D')$ is generated by global sections since the only possible base points are x_{g-2} and x_{g-1} but if either were a base point then some $(g-2)$ of the $\{x_i\}$ would lie in the $(g-4)$-plane $\Lambda_{D'}$ spanned by D'. Let $V = H^0(\Omega(-D'))/H^0(\Omega(-D))$ and let $\widetilde{v_E} : H^0(\Omega(-D')) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega \otimes \mathcal{O}_E$ be the natural map defined by evaluating sections of $\Omega(-D')$ on E. Let $v_E : V \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega \otimes \mathcal{O}_E$ be the induced map. As effective divisors D and D' span hyperplanes λ_D and $\lambda_{D'} \subset \mathbb{P}^{g-1}$ which in particular are the subspaces

$$\Lambda_D = \mathbb{P}(W_D),\ \Lambda_{D'} = \mathbb{P}(W_{D'}) \subset \mathbb{P}(H^0(\Omega)).$$

Then the following commutes.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & V \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r, "v_E"] \arrow[d] & \Omega \otimes \mathcal{O}_E \arrow[d] & \\
0 \arrow[r] & \Sigma_D \arrow[r] \arrow[d] & W_D \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r, "u_D"] \arrow[d] & \Omega \otimes \mathcal{O}_D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma_{D'} \arrow[r] & W_{D'} \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d] & \Omega \otimes \mathcal{O}_{D'} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

If $s \in H^0(\Omega(-D'))$ is some section which does not vanish on D then s cannot vanish at x_{g-2} or x_{g-1}. So $\widetilde{v_E}$ and therefore v_E are surjective. Since $\dim_{\mathbb{C}} V = 1$ implies that $\ker v_E \cong \mathcal{O}_X(-E)$ the following is exact

$$0 \to \mathcal{O}_X(-x_{g-2} - x_{g-1}) \to \Sigma_D \to \Sigma_{D'} \to 0.$$

Finally since D' is composed of a pair of linearly independent points spanning a line $\Lambda_{D'}$

$$\Sigma_{D'} = \mathcal{O}_X(-x_1) \oplus \mathcal{O}_X(-x_2).$$

$\square$

Now the proof of Petri's theorem can proceed as in [GL85].

Theorem B.5.3 ([GL85, page 2]). *Let X be a non-hyperelliptic, smooth, irreducible, projective complex curve of genus $g \geqslant 4$. Suppose A is a degree 4 line bundle on X with $h^0(A) = 2$ such that A and $\omega_X \otimes A^\vee$ are generated by global sections. The the* *homogeneous ideal of X in its canonical embedding $I_{X/\mathbb{P}^{g-1}}$ is generated by forms of degree* 2.

Proof. By the exactness of B.5.1 the following is exact

$$0 \to \bigwedge^2 (\Sigma_D^\vee) \to \bigwedge^2 Q_\Omega \to \Sigma_D^\vee \otimes \Omega(-D) \to 0.$$

The exactness of Lemma B.5.2 implies that

$$0 \to \bigwedge^2 (\mathcal{O}_X(x_1) \oplus \mathcal{O}_X(x_2)) \to \bigwedge^2 \Sigma_D^\vee \to \mathcal{O}_X(x_1+x_{g-2}+x_{g-1}) \oplus \mathcal{O}_X(x_2+x_{g-2}+x_{g-1}) \to 0$$

and

$$0 \to \Omega(-D + x_1) \oplus \Omega(-D + x_2) \to \Sigma_D^\vee \otimes \Omega(-D) \to \Omega(-D + x_{g-2} + x_{g-1}) \to 0$$

are exact. Finally since $g \geqslant 4$ all of the divisors in two previous exact sequences above are properly contained in D so each has a unique section and

$$h^0 \left(\bigwedge^2 \Sigma_D^\vee \right) \leqslant \binom{2}{2} + (g-3).$$

Then since $h^0(\Omega(-D+x_i)) = 2$ for each i but $h^0(\Omega(-D+x_{g-2}+x_{g-1})) = h^0(\Omega(-D')) = 3$ it follows that

$$h^0(\Sigma_D^\vee \otimes \Omega(-D)) \leqslant 2(g-3) + 3.$$

By the exactness of

$$0 \to \bigwedge^2 \Sigma_D^\vee \to \bigwedge^2 Q_\Omega \to \Sigma_D^\vee \otimes \Omega(-D) \to 0,$$

conclude that

$$h^0\left(\bigwedge^2 Q_\Omega\right) \leqslant \binom{2}{2} + 3(g-3) + 3 = \binom{g}{2}.$$

With the bound we have just obtained we see that Lemma B.4.5 applies, and we conclude that the canonical ideal $I_{X/\mathbb{P}^{g-1}}$ is generated by quadrics. $\square$