RESEARCH STATEMENT

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1. Section Rings of Stacky Curves

1.1. **Background and Context.** Computing the section ring of schemes such as curves and surfaces is a problem in algebraic geometry dating back at least as early as Petri's famous theorem that every genus five canonical, non-trigonal curve which is not a smooth plane quintic is the complete intersection of three smooth quadrics in $\mathbb{P}^4$ (see e.g. [GL85] for a formal statement and proof).

For readability we collect some definitions and notation.

Let $\mathscr{X}$ denote a **tame, separably rooted stacky curve**, where we mean **stacky curve** in the sense of [VZB22, Section 5.2]. If D is a Weil divisor on $\mathscr{X}$, then a ring of the form

$$R_D = \bigoplus_{d \geqslant 0} H^0(\mathscr{X}, dD)$$

is called a **section ring** of D, a **canonical ring** in the case when $D = K_{\mathscr{X}}$ is a canonical divisor on $\mathscr{X}$, or a **log canonical ring** if $D = K_{\mathscr{X}} + \Delta$ for some effective divisor Δ on $\mathscr{X}$ supported at distinct points. We say an effective divisor Δ on $\mathscr{X}$ is a **log divisor** if it is given as a sum of distinct points of $\mathscr{X}$. If x is a point on $\mathscr{X}$ has a nontrivial stabilizer (the algebraic spaces representing the functors $\underline{\mathrm{Isom}}(P_i, P_i)$) of order e, then we say x is a **stacky point** of $\mathscr{X}$. Note that [VZB22] specifies that a log divisor is supported at distinct non-stacky points, but we drop this assumption in this document.

Let X denote the **coarse space** of $\mathscr{X}$ and let K_X be a canonical divisor on X. If $\mathscr{X}$ has stacky points $P_1, \cdots, P_r$ of orders $e_1, \cdots, e_r$ respectively, we have a linear equivalence

$$K_{\mathscr{X}} \sim K_X + R = K_X + \sum_{i=1}^{r} \left(1 - \frac{1}{e_i}\right) P_i.$$

If $\mathscr{X}$ is a stacky curve of genus g with stacky points $P_1, \cdots, P_r$ and Δ is a log divisor supported at δ distinct points of $\mathscr{X}$, we call the tuple $(g; e_1, \cdots, e_r; \delta)$, the **signature** of the **log stacky curve** formed by the pair $(\mathscr{X}, \Delta)$. Let $\Omega^1_{\mathscr{X}}$ denote the sheaf of holomorphic differentials on a stacky curve $\mathscr{X}$.

Let $\alpha, \beta \in \mathbb{Q}_{\geqslant 0}$. We will discuss how to use the monoids formed by the lattice $\mathbb{Z}^2$ and the angle between α and $-\beta$:

$$M = \{(d, c) \in \mathbb{Z}^2 : -d\beta \leqslant c \leqslant d\alpha\}$$

to determine bases for section rings of divisors of form $D = \alpha(\infty) + \beta(P)$ on genus 0 and 1 curves.

An example of a canonical ring of a log stacky curve is the algebra of modular forms for Γ some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. We use the following terminology to describe this example. Let $\mathcal{H}$ denote the upper half-plane of the complex numbers $\mathbb{C}$. Let $\Gamma \leqslant \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group with finite coarea. There is an action by $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ by fractional linear transformations. The **cusps** of Γ are the set of Γ-equivalence classes of

$$C(\Gamma) = \{z \in \mathbb{P}^1(\mathbb{Q}) : \gamma z = z \text{ for some } \gamma \text{ with } |\operatorname{tr}(\gamma)| = 2\}.$$

A **modular form** for Γ of **weight** $k \in \mathbb{Z}_{\geqslant 0}$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ such that

$$f(\gamma z) = (cz+d)^k f(z), \quad \text{for all } \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and such that $\lim_{z \to c} f(z)$ exists for all cusps c of Γ.

If k is even, then we have $\dfrac{d}{dz}\left(\dfrac{az+b}{cz+d}\right) = \dfrac{1}{(cz+d)^2}$ since $ad - bc = 1$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. So a given modular form f of weight k for Γ satisfies the transformation rule $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$ if and only if

$$f(\gamma z) d(\gamma z)^{\otimes k/2} = f(z) dz^{\otimes k/2}$$

for all $\gamma \in \Gamma$. We say such differentials $f dz^{\otimes k/2}$ **descend to the modular curve**. We have an isomorphism between modular forms of weight k and differentials $M_k(\Gamma) \xrightarrow{\sim} H^0(X, \Omega^1(\Delta)^{\otimes k/2})$, given by $f(z) \mapsto f(z)dz^{\otimes k/2}$, where X is the coarse space of the modular curve associated to a compactification of $\Gamma \backslash \mathcal{H}$, and Δ is the log divisor of the cusps of Γ, and we leave the thorough treatment of this compactification and log divisor to e.g. [VZB22, Section 6.2].

1.2. **Existing Theory of Section Rings for Stacky Curves.** The cutting edge of this theory are recent techniques to compute the canonical ring of a log stacky curve. The main theorem [VZB22, Theorem 8.3.1] is an inductive result which computes the canonical rings of a given log stacky curve $(\mathscr{X}, \Delta)$ in terms of its signature $(g; e_1, \cdots, e_r; \delta)$ with $r \geqslant 1$ when $g \geqslant 2$, $g = 1$ and $r + 2\delta \geqslant 2$, or when $g = 0$ and $\delta \geqslant 2$, in terms of canonical rings with fewer stacky points, i.e. with signature $(g; e_1, \cdots, e_{r-1}; \delta)$. It is hard to overstate the importance or breadth of this result. My work on stacks deals with a generalization of this theory along the lines of the paper [O'D15].

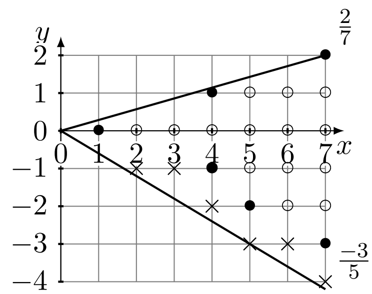
The generalization which O'Dorney considers is the case of a section ring for D, some divisor on $\mathbb{P}^1$ with coefficients in $\mathbb{Q}$. Since D need not correspond to any canonical or log canonical divisor, his ideas are not phrased in terms of signatures like [VZB22]. O'Dorney's main results are [O'D15, Theorems 6 and 8] which compute explicitly the section ring of a 2-point $\mathbb{Q}$-divisor on $\mathbb{P}^1$, and give bounds for the degrees of generators in the general n-point case respectively. In the former, given a divisor $D = \alpha P + \beta Q$ on $\mathbb{P}^1$ with $\alpha, \beta \in \mathbb{Q}$ and $\alpha + \beta \geqslant 0$, O'Dorney find generators and relations in terms of the best lower and upper approximations to α and $-\beta$ respectively. One begins from a common best approximation c_0/d_0, a rational number with minimal denominator such that $-\beta \leqslant c_0/d_0 \leqslant \alpha$. Then the best approximations with greater denominators index generators and give information on relations, all of which may be read off from the monoid M. In [O'D15, Theorem 8], the least common multiples of the denominators of a general $\mathbb{Q}$-divisor $D = \sum_i \alpha_i P_i$ give bounds for the degrees of generators and relations for the section ring of D.

1.3. **My Work on Section Rings for Stacky Curves.** In joint work with Evan O'Dorney and Michael Cerchia [CFO23] we consider canonical rings of $\mathbb{Q}$-divisors on stacky curves of genus 1. Many of the techniques, such as the consideration of monoid algebras M and the use of best approximations from [O'D15] apply to this situation, but there are several subtleties for stacky elliptic curves which are not present in the genus 0 case.

In particular, whereas on $\mathbb{P}^1$ any three points may be mapped to any three others with some automorphism, there are torsion points on elliptic curves. This changes the nature of the Riemann-Roch spaces, since for example there are no functions with single simple poles. According to whether or not a given $\mathbb{Q}$-divisor D is supported at torsion points, there are significant changes to the monoids M and the indices of generators for the minimal presentation of the canonical ring R_D therefore depend on whether D is supported at torsion points or not.

Example 1.1. *Suppose that* $D = \frac{2}{7}(\infty) + \frac{3}{5}P$ *is an effective divisor on a genus* 1 *curve* E, *where* P *is non-torsion on* E. *Then the ring* R_D *is generated by functions in degrees* $1, 4, 4, 5, 7, 7$.

$$M = \left\{(d, c) \in \mathbb{Z}^2 : -d\left(\frac{3}{5}\right) \leqslant c \leqslant d\left(\frac{2}{7}\right)\right\}.$$



Shaded-in points of M *denote generators for* R_D *in the case when* D *is not supported at any torsion points. Open dots indicate functions on* E *which correspond to linear combinations of shaded dots in* M. *Finally, the* $\times$*-dots in the diagram indicate points in* M *which would correspond to generators for* M, *and hence* R_D, *if* P *were* d*-torsion on* E, *where* d *denotes the* x*-coordinate of such a point in* M. *Such square points do not correspond to generators for* M *when* P *is non-torsion.*

We have results for several cases. We can describe explicitly the generators for the section ring of a $\mathbb{Q}$-divisor supported at exactly one point with [CFO23, Theorem 2.3] and can describe initial terms in the ideal of relations for the ring in [CFO23, Theorem 2.7]. In the effective case of 2-point $\mathbb{Q}$-divisors, say $D = \alpha^{(1)}(P^{(1)}) + \alpha^{(2)}(P^{(2)})$ is some divisor on an elliptic curve C supported on two points $P^{(i)}$. We may assume that $\alpha^{(1)} \geqslant \alpha^{(2)}$ since the roles of the points $P^{(i)}$ may be interchanged. We explicitly describe the generators for the section ring R_D in [CFO23, Theorem 3.2] and can give initial terms for the ideal of relations for R_D with [CFO23, Theorem 3.5]. We aim to give a bound for the maximal degree of generators in a minimal presentation of R_D in the general case of a divisor supported at n points.

2. Drinfeld modular forms

2.1. **Background.** The Drinfeld setting is a function-field analog of the classical number-field theory which is the usual for an arithmetic geometer. There are versions of some notable features of classical number theory, for example elliptic and modular curves, and modular forms. The Drinfeld setting also features analogs of class-field theory, such as the function-field version of the Kronecker-Weber theorem and the theory of Galois representations of K which motivated Drinfeld's original work. More recent work of Goss and Gekeler defines more objects familiar to an arithmetic geometer in the Drinfeld setting. In his 1986 manuscript [Gek86, Page XIII] asks for an arithmetic interpretation of Drinfeld modular forms for congruence subgroups, in particular in terms of generators and relations for the algebra of Drinfeld modular forms.

We introduce this theory by collection definitions and notation here for ease of reading.

In place of $\mathbb{Z}$, consider the polynomial ring $A = \mathbb{F}_q[T]$ for q a power of an odd prime, for $\mathbb{Q}$ we take the fraction field $K = \mathbb{F}_q(T)$, the completion at the infinite place is denoted $K_\infty = \mathbb{F}_q((1/T))$. The algebraic closure of K_∞ is infinite dimensional over K_∞ and is not complete with respect to the unique extension of the degree valuation, but the completed algebraic closure C is both complete and algebraically closed.

The Drinfeld setting version of the upper half-plane is the rigid analytic space (in the sense of [FvdP04]) $\Omega = C - K_\infty$. In the Drinfeld setting, we have an action of $\mathrm{GL}_2(A)$ on Ω by fractional linear transformations, and we note that $\det : \mathrm{GL}_2(A) \to \mathbb{F}_q^\times$. Let $N \in A$ be some non-constant, monic polynomial. Let $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$ be the subgroups of $\mathrm{GL}_2(A)$ containing the matrices of forms

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}, \text{ and } \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ modulo } N$$

respectively. We say a **congruence subgroup** is some subgroup Γ of $\mathrm{GL}_2(A)$ containing $\Gamma(N)$ for some N, and we call the N of least degree the **conductor** of Γ. As in [Bre16], for Γ a congruence subgroup let

$$\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \text{ is the square of some element of } \mathbb{F}_q^\times\}.$$

In analogy with the parameter $q = e^{2\pi i z}$ at ∞ in the classical setting, we define a parameter at ∞ in the Drinfeld setting. First we fix a **Carlitz period** $\overline{\pi} \in K_\infty(\sqrt[q-1]{-T})$ (see [Car38] for more exposition). Then, for $e_A(z)$ an exponential function (see e.g. [GR96, (2.7.2)] for a definition), we define a **parameter at** ∞, $u(z)$, by

$$u(z) = \frac{1}{\overline{\pi} e_A(z)} = \overline{\pi}^{-1} \sum_{a \in A} \frac{1}{z+a}.$$

Unlike the classical setting, the differential dz has a double pole at ∞ in the Drinfeld setting, since $\dfrac{de_A(z)}{dz} = 1$, so that $\dfrac{du}{u^2} = -\overline{\pi} dz$.

Definition 2.1. [Gek86, Definition (3.1)] *Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup. A **modular form** of **weight** $k \in \mathbb{Z}_{\geqslant 0}$ and **type** $l \in \mathbb{Z}/((q-1)\mathbb{Z})$ is a holomorphic function $f : \Omega \to C$ such that*

(1) *$f(\gamma z) = \det(\gamma)^{-l}(cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & b \end{pmatrix} \in \Gamma$, and*
(2) *f is holomorphic at the cusps of Γ.*

Write $M_{k,l}(\Gamma)$ for the finite-dimensional C-vector space of Drinfeld modular forms for a congruence subgroup $\Gamma \leqslant \mathrm{GL}_2(A)$ with weight k and type l (see [Gos80] for a proof this space is finite dimensional). The algebra $M(\Gamma)$ of modular forms graded coarsely by weight and finely by type, i.e.

$$M(\Gamma) = \bigoplus_{\substack{k \geqslant 0 \\ l \pmod{q-1}}} M_{k,l}(\Gamma),$$

since $M_{k,l} \cdot M_{k',l'} \subset M_{k+k',l+l'}$.

The weight and type of a Drinfeld modular form are not independent: if there is a non-zero modular form f of weight $k \in \mathbb{Z}_{\geqslant 0}$ and $l \in \mathbb{Z}/(q-1)\mathbb{Z}$, then $k \equiv 2l \pmod{q-1}$. Indeed, if $\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ for some $\alpha \in \mathbb{F}_q^\times$ and if f is non-zero modular for Γ of weight k and type l then

$$f(\gamma z) = f\left(\frac{\alpha z}{\alpha}\right) = \alpha^k \alpha^{-2l} f(z) = f(z),$$

so $\alpha^k = \alpha^{2l}$ in $\mathbb{F}_q^\times$ and we conclude $k \equiv 2l \pmod{q-1}$.

Drinfeld modular forms have series expansions in terms of this parameter $u(z)$ at ∞ analogous to q-series for classical modular forms, and likewise we say a form with coefficient $a_0 = 0$ is a **cusp form**.

Drinfeld modules of rank 2, the function-field analogs of elliptic curves, have likewise to elliptic curves, a moduli space which can be described by quotients. A quotient of the compactification $\Omega \cup \mathbb{P}^1(K)$ by a congruence subgroup Γ is a smooth, rigid analytic space of dimension 1. This quotient is algebraic in the sense that there is a canonical isomorphism to the underlying (rigid) analytic space of a smooth irreducible projective curve over C. That is, the C-points of an **algebraic Drinfeld modular curve** X_Γ are in bijection with the **analytification** X_Γ^{an}, whose points are defined by the quotient of $\Gamma\backslash\Omega \cup \mathbb{P}^1(K)$. It is well-known in the subject (see both[Lau96] and [Gek86]) that the moduli space of rank r Drinfeld modules with Γ level structure is an algebraic Deligne-Mumford stack. In the rank 2 case we denote this stack, which we will call the **(stacky) Drinfeld modular curve**, by $\mathscr{X}_\Gamma$. The **cusps** of $\mathscr{X}_\Gamma$ are the orbits $\Gamma\backslash\mathbb{P}^1(K)$.

Goss establishes in [Gos80] a correspondence between Drinfeld modular forms for a congruence subgroup Γ and differentials on the modular curve $\Gamma(\backslash\Omega \cup \mathbb{P}^1(K))$. In the same paper, Goss shows

$$M_0(\mathrm{GL}_2(A)) = \bigoplus_{k \geqslant 0} M_{k,0}(\mathrm{GL}_2(A)) = C[g, \Delta]$$

for some modular forms g and Δ. Gekeler shows in [Gek88]

$$M(\mathrm{GL}_2(A)) = \bigoplus_{\substack{k \geqslant 0 \\ l \pmod{q-1}}} M_{k,l}(\mathrm{GL}_2(A)) = C[g, h],$$

where is the modular form from Goss' result, and h is another Drinfeld modular form related to Δ. See [Bre16] for an exposition on h.

2.2. **My Work on Drinfeld Modular Forms.** Because of Goss' connection between forms and differentials, the algebra of Drinfeld modular forms may be thought of as a section ring for the sheaf of differentials as in the classic setting. One might hope to use geometric invariants of (stacky) Drinfeld modular curves and a program like [VZB22] to compute the section rings and then recover the algebra of Drinfeld modular forms. Here are some subtleties which complicate this procedure.

First, if we define a log divisor Δ supported at the cusps of $\mathscr{X}_\Gamma$, then Δ may be supported at stacky points. Indeed, the subgroup of upper triangular matrices in Γ, which stabilizes $\mathbb{P}^1(K)$, may contain matrices of form $\begin{pmatrix} \alpha & * \\ 0 & \alpha' \end{pmatrix}$ for some $\alpha, \alpha' \in \mathbb{F}_q^\times$, and this group is larger than the generic stabilizers of points in $\mathscr{X}$ formed by matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, where $\alpha \in \mathbb{F}_q^\times$.

Second, if $f \in M_{k,l}$ is a non-zero modular form, then the differential $f(dz)^{\otimes k/2}$ may not descend to the modular curve X_Γ^{an} since the differential is not necessarily Γ-invariant. We have

$$\begin{aligned} f(\gamma z) d(\gamma z)^{\otimes k/2} &= (cz+d)^k (\det\gamma)^{-l} \frac{\det\gamma^{k/2}}{(cz+d)^k} f(z) dz^{\otimes k/2} \\ &= (\det\gamma)^{\otimes l - k/2} f dz^{\otimes k/2}, \end{aligned}$$

where since $\det\gamma \in \mathbb{F}_q^\times$, it need not be that $(\det\gamma)^{\otimes l-k/2} = 1$, unless $\det\gamma$ is the square of some element in $\mathbb{F}_q^\times$. We may get around this problem by considering only Γ with $\det(\gamma)$ a square in $\mathbb{F}_q^\times$ for all $\gamma \in \Gamma$, or by comparing modular forms for a general Γ to the modular forms for Γ_2.

My main results in the Drinfeld setting so far are the following:

Theorem 2.2. [Fra23] *Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices in $\mathrm{GL}_2(A)$ and such that $\det(\gamma) \in (\mathbb{F}_q^\times)^2$ for all $\gamma \in \Gamma$. Let Δ be the (Weil) divisor of cusps of the*

Drinfeld modular curve $\mathscr{X}_\Gamma$ *with rigid analytic coarse space* $X_\Gamma^{an} = \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$. *There is an isomorphism of graded rings*

$$M(\Gamma) \cong R(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(2\Delta)),$$

where $\Omega^1_{\mathscr{X}_\Gamma}$ *is the sheaf of differentials on* $\mathscr{X}_\Gamma$. *The isomorphism is given by isomorphisms*

$$M_{k,l}(\Gamma) \to H^0(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(2\Delta)^{\otimes k/2})$$

of components given by $f \mapsto f(dz)^{\otimes k/2}$.

Theorem 2.3. [Fra23] *Let* $\Gamma \leqslant \mathrm{GL}_2(A)$ *be a congruence subgroup containing the diagonal matrices in* $\mathrm{GL}_2(A)$. *Let* $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$. *Then* $M(\Gamma) \cong M(\Gamma_2)$, *with*

$$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$

on each graded piece, where l_1, l_2 *are the two solutions to* $k \equiv 2l \pmod{q-1}$.

Remark 1. If l_1, l_2 are the two solutions to $k \equiv 2l \pmod{q-1}$, then we have $M_{k,l_1}(\Gamma) = M_{k,l_2}(\Gamma)$.

Theorem 2.4. [Fra23] *Let* $\Gamma \leqslant \mathrm{GL}_2(A)$ *be a congruence subgroup. Let* $\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$. *Suppose that* $\Gamma_1 \leqslant \Gamma' \leqslant \Gamma$ *for some congruence subgroup* Γ'. *Then as algebras*

$$M(\Gamma) = M(\Gamma'),$$

and each component $M_{k,l}(\Gamma')$ *is some direct sum of components* $M_{k,l'}(\Gamma)$ *for some nontrivial* l'.

3. Connections and Future Work

Thanks to the connection between modular forms and differentials, it is possible to compute algebras of Drinfeld modular forms in terms of section rings for divisors on Drinfeld modular curves in some cases by using geometric invariants of stacks as in [VZB22], [O'D15] and [CFO23]. We can also make sense of a Deligne-Mumford rigid analytic stacky curve by comparing the theories in [BN05], [EGH23] and [PY16], so we can pose a series of conjectures regarding their uniformization and ultimately perhaps classification in a similar way to [BN05].

There are many Drinfeld modular curves which have higher genera than 0 and 1, and even with those small genera, many such curves have signatures where existing techniques for computing section rings cannot give explicit generators and relations for the algebra of modular forms. There are some existing results which approach Gekeler's problem, such as Cornelissen's [Cor97, Theorem (3.3)], which gives the algebra of modular forms for $\Gamma(\alpha T + \beta)$ and we know the algebra of Drinfeld modular forms for $\Gamma_0(T)$ from [DK23, Theorem (4.4)]. Even by the date of these more recent papers, the generalization to the algebra of modular forms for $\Gamma_0(N)$ for any level N, all $\Gamma_1(N)$, and all higher level ($\deg(N) \geqslant 2$) $\Gamma(N)$ examples seem to be wide open.

Furthermore, our main theorems about the algebra of modular forms for Γ applies to congruence subgroups containing $\Gamma_0(N)$ since they rely on the hypothesis that Γ contains the diagonal matrices in $\mathrm{GL}_2(A)$. Both our assumption and the geometric invariants we observe limit the possible examples which one can immediately compute. It is worth considering algebras of Drinfeld modular forms for congruence subgroups $\Gamma' \leqslant \Gamma$, where $\Gamma_1 \leqslant \Gamma' \leqslant \Gamma$ as in Theorem 2.4. In a similar vein, algebras of Drinfeld modular forms for $\mathrm{SL}_2(A)$ and the congruence subgroups of $\mathrm{SL}_2(A)$ are accessible and include examples of Drinfeld modules with higher level and higher genera modular curves thanks to results in [GvdP80].

This indicates several directions for future work on stacks as well. In the computation of section rings on stacks, one might hope to use the monoid algebras in [VZB22], [O'D15] and [CFO23] to give more explicit minimal presentations of general n-point $\mathbb{Q}$-divisors on stacky curves of genus 1. This theory also ought to give some results for $\mathbb{Q}$-divisors on higher genera curves. Eventually the inductive presentation of the canonical ring for log stacky curves in all genera in the spirit of the main result from [VZB22] should be generalized to a wider class of stacks such as non-tame cases, though certainly the combinatorics of the presentation of such rings will vary greatly depending on context.

References

[BN05] K. Behrend and B. Noohi, *Uniformization of deligne-mumford curves*, 2005.

[Bre16] Florian Breuer, *A note on Gekeler's h-function*, Arch. Math. (Basel) **107** (2016), no. 4, 305–313. MR 3552209

[Car38] Leonard Carlitz, *A class of polynomials*, Trans. Amer. Math. Soc. **43** (1938), no. 2, 167–182. MR 1501937

[CFO23] Michael Cerchia, Jesse Franklin, and Evan O'Dorney, *Section rings of $\mathbb{Q}$-divisors on genus-1 curves*, 2023, `https://arxiv.org/abs/2312.15128`.

[Cor97] Gunther Cornelissen, *Drinfeld modular forms of weight one*, Journal of Number Theory **67** (1997), no. 2, 215–228.

[DK23] Tarun Dalal and Narasimha Kumar, *The structure of Drinfeld modular forms of level $\Gamma_0(T)$ and applications*, Journal of Algebra **619** (2023), 778–798.

[EGH23] Matthew Emerton, Toby Gee, and Eugen Hellmann, *An introduction to the categorical p-adic langlands program*, 2023.

[Fra23] Jesse Franklin, *The geometry of Drinfeld modular forms*, 2023, `https://arxiv.org/abs/2310.19623`.

[FvdP04] Jean Fresnel and Marius van der Put, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser Boston, Inc., Boston, MA, 2004. MR 2014891

[Gek86] Ernst-Ulrich Gekeler, *Drinfeld modular curves*, Lecture Notes in Mathematics, vol. 1231, Springer-Verlag, Berlin, 1986. MR 874338

[Gek88] ———, *On the coefficients of Drinfeld modular forms*, Invent. Math. **93** (1988), no. 3, 667–700. MR 952287

[GL85] Mark Green and Robert Lazarsfeld, *A simple proof of Petri's theorem on canonical curves*, Geometry today (Rome, 1984), Progr. Math., vol. 60, Birkhäuser Boston, Boston, MA, 1985, pp. 129–142. MR 895152

[Gos80] David Goss, *The algebraist's upper half-plane*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), no. 3, 391–415. MR 561525

[GR96] E.-U. Gekeler and M. Reversat, *Jacobians of Drinfeld modular curves*, J. Reine Angew. Math. **476** (1996), 27–93. MR 1401696

[GvdP80] Lothar Gerritzen and Marius van der Put, *Schottky groups and mumford curves*, Lecture Notes in Mathematics, vol. 817, Springer Berlin, Heidelberg, 1980.

[Lau96] Gérard Laumon, *Cohomology of Drinfeld modular varieties. Part I*, Cambridge Studies in Advanced Mathematics, vol. 41, Cambridge University Press, Cambridge, 1996, Geometry, counting of points and local harmonic analysis. MR 1381898

[O'D15] Evan O'Dorney, *Canonical rings of $\mathbb{Q}$-divisors on $\mathbb{P}^1$*, Annals of Combinatorics **19** (2015), no. 4, 765–784.

[PY16] Mauro Porta and Tony Yue Yu, *Higher analytic stacks and GAGA theorems*, Adv. Math. **302** (2016), 351–409. MR 3545934

[VZB22] John Voight and David Zureick-Brown, *The canonical ring of a stacky curve*, Mem. Amer. Math. Soc. **277** (2022), no. 1362, v+144. MR 4403928