Geometry of Drinfeld Modular Forms

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The Drinfeld Setting

q - a power of an odd prime.
K - the function field of some smooth, connected, projective curve over a field of characteristic q, e.g. $\mathbb{P}^1$

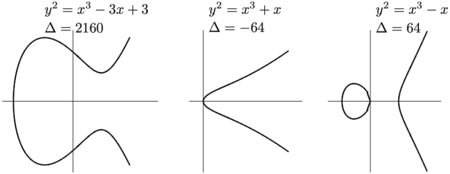
Classical Setting	Function Field
$\mathbb{Z}$	$A \stackrel{def}{=} \mathbb{F}_q[T]$
$\mathbb{Q}$	$K \stackrel{def}{=} Frac(A) = \mathbb{F}_q(T)$
$\mathbb{R}$	$K_\infty \stackrel{def}{=} \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$
$\mathbb{C}$	$C \stackrel{def}{=} \widehat{\overline{K_\infty}}$
$\mathcal{H} = \{a + bi \in \mathbb{C} : b > 0\}$	$\Omega \stackrel{def}{=} C - K_\infty$
$\mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}$	$\mathrm{GL}_2(A) \setminus \Omega$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

Elliptic Curves and Drinfeld Modules

Elliptic Curves

An **elliptic curve** is (analytically) a torus$/\mathbb{C}$, i.e. a lattice quotient $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$ for $z \in \mathcal{H}$;
or (algebraically) a curve defined by:
$E : y^2 = x^3 + A(z)x + B(z)$



[Sil09, Figure 3.1]

Drinfeld Modules

Consider the rank 2 lattice $\Lambda_z = \overline{\pi}(zA + A) \subset C$.The associated **Drinfeld module of rank** 2 is given by

$$\varphi^z(T) = TX + g(z)X^q + \Delta(z)X^{q^2},$$

the image of a ring homomorphism $\varphi^z : A \to C\{X^q\}$ where $C\{X^q\}$ is the non-commutative ring of $\mathbb{F}_q$-linear polynomials$/C$.

Moduli Problems

Let $\Gamma^1 \leq \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma \leq \mathrm{GL}_2(A)$ be subgroups.

$$\left(\begin{array}{c} \text{quotient spaces} \\ \Gamma^1 \setminus \mathcal{H} \text{ (resp. } \Gamma \setminus \Omega) \end{array} \right) \overset{\text{classify}}{\leftrightarrow} \left(\begin{array}{c} \text{families of elliptic curves} \\ \text{(resp. Drinfeld modules of rank 2)} \\ \text{which have torsion info} \end{array} \right)$$

For example,

$$\Gamma_0(N) \overset{def}{:=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{N} \right\}$$

corresponds to the **moduli space** of

$$\begin{cases} \text{elliptic curves} \\ \text{Drinfeld modules of rank 2} \end{cases} \quad \text{with an } N\text{-torsion subgroup.}$$

Classical Modular Forms & Curves

Algebraic Modular Curve		Analytic Moduli Space
$\mathscr{X}_\Gamma$	GAGA	$\Gamma \setminus \mathcal{H}^*$
Deligne-Mumford (stacky) curve	$\leftrightarrow$	Compact Riemann surface (orbifold)

Definition ([DS05, 1.1.2])

A map $f : \mathcal{H} \to \mathbb{C}$ is a **modular form of weight** $k \in \mathbb{Z}$ for $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ if

1. f is holomorphic on $\mathcal{H}$ and at cusps of Γ; and
2. $f(\gamma z) = (cz + d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $z \in \mathcal{H}$.

We know (e.g. [VZB22, Chapter 6])

$$M(\Gamma) \overset{def}{:=} \bigoplus_{k \geq 0} M_k(\Gamma) \xrightarrow{\sim} \bigoplus_{k \geq 0} H^0(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(\Delta)^{\otimes k/2}) \overset{def}{=:} R(\mathscr{X}_\Gamma, \Delta),$$

$$f \mapsto f dz^{\otimes k/2}$$

"Ingredients"

1. **(Log) Stacky Curve** $(\mathscr{X}, \Delta)$ ([LRZ16, Def 2.1] and [VZB22, Ch 4])
   - a "nice" scheme $X/\overline{\mathbb{K}}$ of dimension 1, together with "fractional" (*stacky*) points $\frac{1}{e_1}P_1, \ldots, \frac{1}{e_r}P_r$ of X with $e_i \in \mathbb{Z}_{\geq 2}$;
   - a *log divisor* is some $\Delta \in \operatorname{Div}(\mathscr{X})$ a sum of distinct points of $\mathscr{X}$
2. **(Ample) Line Bundle**
   - e.g. $K_{\mathscr{X}} \sim \Omega^1_{\mathscr{X}}$ or $K_{\mathscr{X}} + \Delta$
   - gives an embedding of $\mathscr{X}$ in projective space
3. **Modular forms "=" Sections** - à la $(f \mapsto fdz^{\otimes k/2})$
4. **GAGA** - equivalences of categories:

$$\begin{pmatrix} \text{algebraic} \\ \text{curves and bundles} \end{pmatrix} \xrightarrow{\cong} \begin{pmatrix} \text{analytic} \\ \text{curves and bundles} \end{pmatrix}$$

Drinfeld Modular Forms & Curves

Algebraic Modular Curve	rigid (stacky) GAGA	Analytic Moduli Space
$\mathscr{X}_\Gamma$ Deligne-Mumford (stacky) curve	$\leftrightarrow$	$\Gamma \setminus (\Omega \cup \mathbb{P}^1(K))$ compact rigid analytic stack

Definition ([Gek86, (3.1)])

Let $\Gamma \leq \mathrm{GL}_2(A)$ be a congruence subgroup. A **modular form** of **weight** $k \in \mathbb{Z}_{\geq 0}$ and **type** $l \in \mathbb{Z}/((q-1)\mathbb{Z})$ is a holomorphic function $f : \Omega \to C$ such that

1. f is holomorphic on Ω and at the cusps of Γ; and
2. $f(\gamma z) = \det(\gamma)^{-l}(cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & b \end{pmatrix} \in \Gamma$.

Geometry of Drinfeld Modular Forms (1/3)

Let q be odd;
Let $\Gamma \leq \mathrm{GL}_2(A)$;
Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma_2} \\ \big\downarrow \\ \mathscr{X}_{\Gamma} \end{array}$$

When we compute the log canonical ring $R(\mathscr{X}_{\Gamma_2}, 2\Delta)$ we get the following result.

Theorem ([Fra23, 6.1])

There is an isomorphism of graded rings

$$M(\Gamma_2) \cong R(\mathscr{X}_{\Gamma_2}, \Omega^1_{\mathscr{X}_{\Gamma_2}}(2\Delta)),$$

given by isomorphisms

$$M_{k,l}(\Gamma_2) \to H^0(\mathscr{X}_{\Gamma_2}, \Omega^1_{\mathscr{X}_{\Gamma_2}}(2\Delta)^{\otimes k/2})$$

of form $f \mapsto f(dz)^{\otimes k/2}$, *where* $k \equiv 2l \pmod{q-1}$.

Geometry of Drinfeld Modular Forms (2/3)

Let q be odd;
Let $\Gamma \leq \mathrm{GL}_2(A)$;
Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma_2} \\ \big\downarrow \\ \mathscr{X}_{\Gamma} \end{array}$$

When we compare the modular forms for Γ and Γ_2 we find the following.

Theorem ([Fra23, 6.2])

We have $M(\Gamma) \cong M(\Gamma_2)$, with

$$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$

on each component,where l_1, l_2 are the solutions to $k \equiv 2l \pmod{q-1}$.

Geometry of Drinfeld Modular Forms (3/3)

Let q be odd;
$\Gamma \leq \mathrm{GL}_2(A)$;
$\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$.
Suppose that $\Gamma_1 \leq \Gamma' \leq \Gamma$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma'} \\ \big\downarrow \\ \mathscr{X}_{\Gamma} \end{array}$$

When we compare the modular forms for Γ and Γ' we find the following generalization of [Fra23, Theorem 6.2].

Theorem ([Fra23, 6.12])

We have $M(\Gamma) \cong M(\Gamma')$, and each component $M_{k,l}(\Gamma')$ is some direct sum of components $M_{k,l'}(\Gamma)$ for some nontrivial l'.

Conclusion

Thank you!
Further details available at arXiv:2310.19623

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