§3.1: Limits

Limit of a function: Let f be a function and let $a, L \in \mathbb{R}$. If:

(1) As x takes values closer and closer (but not equal) to a on "both sides" of a (positive & negative), the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to L; and

(2) The value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a;

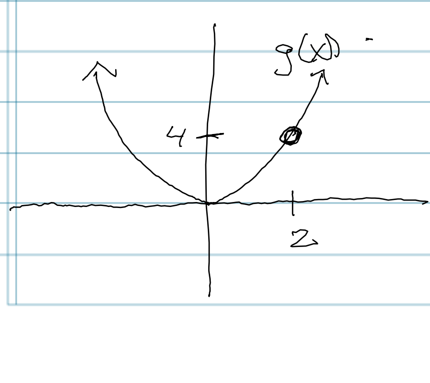
Then L is the limit of $f(x)$ as x approaches a, written:

$$\boxed{\lim_{x \to a} f(x) = L}$$

e.g. Find the limit:

(A) $\lim_{x \to 2} g(x)$, where $g(x) = \dfrac{x^3 - 2x^2}{x-2} = x^2 \dfrac{(x-2)}{(x-2)}$

- Since $g(2)$ does not exist (there is a hole at $g(2)$ $(x=2)$) we cannot directly consider $g(2)$. However, when we cancel each factor of $(x-2)$ we get
$g(x) = x^2$; $x \neq 2$. This graph follows the reduced form of $g(x)$ with a hole at $x=2$.

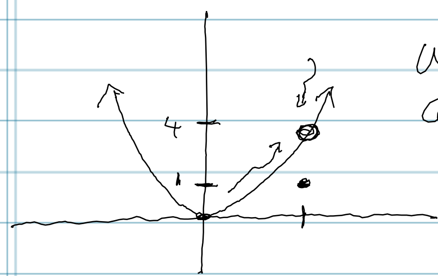


Now, we see as $x \to 2$ from either side, the graph of $g(x)$ approaches $2^2 = 4$. Thus

$$\lim_{x \to 2} g(x) = 2^2 = \boxed{4}$$

(B) Let $h(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$; Find $\lim_{x \to 2} h(x)$

- This function is very close to our last function, g, however instead of a hole at $x=2$, the value at 2 is defined as $h(2) = 1$



We see the function is moving toward the same point as before, even though $g(2) = 1$

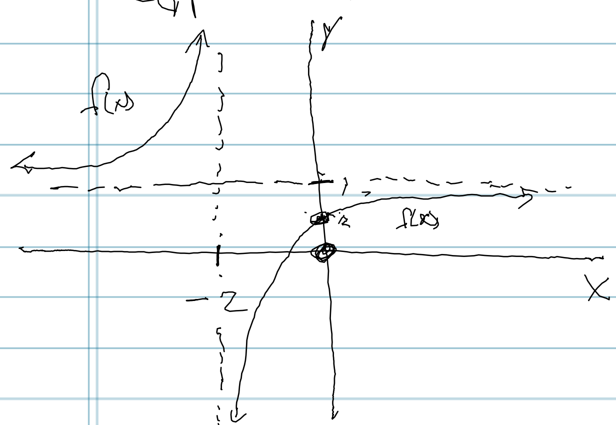
$$\lim_{x \to 2^-} h(x) = \lim_{x \to 2^+} h(x) = 2^2 = \boxed{4}$$

(C) find $\lim_{x \to -2} f(x)$ of $f(x) = \dfrac{3x+2}{2x+4}$

- we must check when the denominator can be zero

$$2x+4=0 \Rightarrow x = \frac{-4}{2} = -2$$

So we must consider behavior of the function as it approaches the vertical asymptote from each direction.



We see as we approach -2 from the left (negative side), then f is getting very large so

$$\lim_{x \to -2^-} f(x) = \infty$$

Similarly, looking from the right

$$\lim_{x \to -2^+} f(x) = -\infty$$

We conclude since

$$\lim_{x \to -2^+} \neq \lim_{x \to -2^-}$$ that $$\lim_{x \to -2} f(x) = \text{DNE}$$

"Does not exist"

Existence of Limits: The limit of f(x) as x approaches a may not exist

(1) If f(x) becomes ∞'ly large in magnitude (positive or negative) as x approaches a from either side we say $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$ (resp.) In either case the limit Does Not Exist (still denote it as $\pm\infty$)

(2) If f(x) becomes infinitely large in magnitude (positive) as x approaches a from one side and ∞'ly large (negative) as x approaches a from the other side, then $\lim_{x \to a} f(x)$ DOES NOT EXIST

(3) If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} = M$ and $L \neq M$ then

$\lim_{x \to a} f(x)$ does NOT Exist

Rules For Limits

Let $A, B \in \mathbb{R}$, f, g are functions ST

$\lim_{x \to a} f(x) = A$ & $\lim_{x \to a} g(x) = B$

(1) If K is a constant ($K \in \mathbb{R}$), then $\lim_{x \to a} K = K$

$$\lim_{x \to a} [K \cdot f(x)] = K \cdot \lim_{x \to a} f(x) = KA$$

(2) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$

("Sum of the limits equals limit of the sums")

(3) $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$

("Product of limits equals limit of products")

(4) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{A}{B}$; If $B \neq 0$

("quotient of limits is limit of quotients if denom $\neq 0$")

(5) If P is a polynomial, then $\lim_{x\to a} P(x) = P(a)$ (polynomials are nice and smooth)

(6) For any real # k, ($k \in \mathbb{R}$): $\lim_{x\to a} (f(x))^k = \left[\lim_{x\to a} f(x)\right]^k = A^k$

(Provided this limit exists)

(7) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if $f(x) = g(x)$, $\forall x \neq a$

(8) $\forall b \in \mathbb{R}_{>0}$, $\lim_{x\to a} b^{f(x)} = b^{\left[\lim_{x\to a} f(x)\right]} = b^A$

(9) For any b such that $0 < b < 1$ or $b > 1$

$\lim_{x\to a}[\log_b f(x)] = \log_b \left[\lim_{x\to a} f(x)\right] = \log_b A$ if $A > 0$

e.g.) Find the Limit

(A) $\lim\limits_{x\to 3} \dfrac{x^2-x-1}{\sqrt{x+1}} \overset{(4)}{=} \dfrac{\lim\limits_{x\to 3} x^2-x-1}{\lim\limits_{x\to 3} \sqrt{x+1}} \underset{(6)}{\overset{(5)}{=}} \dfrac{\overset{9-3-1}{\overset{\shortparallel}{5}}}{\sqrt{\lim\limits_{x\to 3} x+1}} \overset{(5)}{=} \boxed{\dfrac{5}{2}}$

(B) $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2} \overset{\text{Try }(5)}{=} \dfrac{2^2+2-6}{2-2} = \dfrac{0}{0}$ = Indeterminate Form

This tells us, we need to manipulate our function (most likely by factoring and cance

$\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2} = \lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}} = \lim\limits_{x\to 2} x+3 \overset{(5)}{=} 2+3 = 5$

(C) $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4} \overset{\text{Try }(5)}{=} \dfrac{\sqrt{4}-2}{4-4} = \dfrac{0}{0}$ Indeterminate form. Can rationalize the numerator or recognize the denominator is a difference of squares

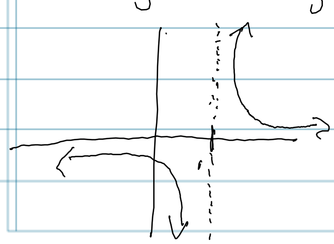
$\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4} = \dfrac{\cancel{(\sqrt{x}-2)}}{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)} = \lim\limits_{x\to 4} \dfrac{1}{\sqrt{x}+2}$

$= \dfrac{1}{\sqrt{4}+2} = \dfrac{1}{2+2} = \boxed{\dfrac{1}{4}}$

(D) $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{(x-1)^3} = \lim\limits_{x\to 1} \dfrac{(x-1)^2}{(x-1)^3} = \lim\limits_{x\to 1} \dfrac{1}{x-1}$

We see there is a singularity at $x=1$ (since $x-1=0 \Rightarrow x=1$)

Looking at the graph:



We see $\lim\limits_{x\to 1^+} \dfrac{1}{x-1} = +\infty$

$\lim\limits_{x\to 1^-} \dfrac{1}{x-1} = -\infty$ $\Rightarrow$ $\boxed{\text{Limit DNE}}$

Finding l's at Infinity:

- If $P(x)$ and $Q(x)$ are polynomials with $Q(x) \neq 0$ and $f(x) = \frac{P(x)}{Q(x)}$ then $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ can be found as:

Steps:

1. Divide $P(x)$ and $Q(x)$ by highest power of x in $Q(x)$
2. Use the rules for limits including the rules for limits at infinity:

$$\lim_{x \to +\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^n} = 0 \;, n > 0$$

to find result of limit in step 1.

e.g. | Find the limit

(A) $\lim_{x \to \infty} \frac{8x+6}{3x-1} = \lim_{x \to \infty} \frac{(8x+6) \cdot \frac{1}{x}}{(3x-1) \frac{1}{x}} = \lim_{x \to \infty} \frac{8 + \frac{6}{x}}{3 - \frac{1}{x}} = \frac{8-0}{3-0} = \boxed{\frac{8}{3}}$

(B) $\lim_{x \to \infty} \frac{3x+2}{4x^3-1} = \lim_{x \to \infty} \frac{\frac{3}{x^2} + \frac{2}{x^3}}{4 - \frac{1}{x^3}} = \frac{0+0}{4-0} = \frac{0}{4} = \boxed{0}$

(C) $\lim_{x \to \infty} \frac{3x^2+2}{4x-3} = \lim_{x \to \infty} \frac{3x + 2/x}{4 - 3/x}$ } we see that factor of x ($3x$ term) will get arbitrarily large

$\Rightarrow \lim_{x \to \infty} \frac{3x^2+2}{4x-3} = \boxed{\infty}$

(D) $\lim_{x \to \infty} \frac{5x^2-4x^3}{3x^2+2x-1} = \lim_{x \to \infty} \frac{5-4/}{3 - \frac{2}{x} - \frac{1}{x^2}}$ } Here the top just gets large and Negative

$\Rightarrow \lim_{x \to \infty} \frac{5x^2-4x^3}{3x^2+2x-1} = \boxed{-\infty}$

§3.2 : Continuity

Continuity at $x=c$

A function f is continuous at a point $x=c$ if the following are satisfied:

(1) $f(c)$ is defined

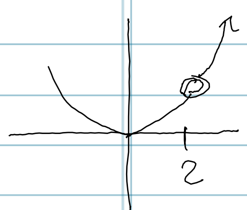
(2) $\lim_{x\to c} f(x)$ exists, and

(3) $\lim_{x\to c} f(x) = f(c)$

If f is NOT continuous at c, it is called discontinuous there

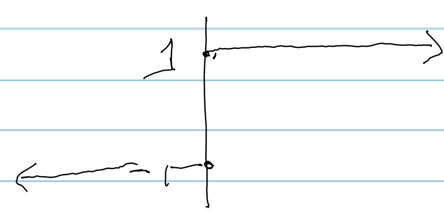
e.g. | Determine if the function is continuous at the indicated x-value

(A) $f(x) = \dfrac{x^3-2x^2}{x-2}$ at $x=2$



$= \dfrac{x^2\cancel{(x-2)}}{\cancel{(x-2)}} = x^2;\ x\neq 2$ i.e. $f(2)$ is undefined, so by (1) f is NOT continuous at 2

(B) $h(x) = \begin{cases} 1, & \text{if } x>0 \\ -1, & \text{if } x\le 0 \end{cases}$



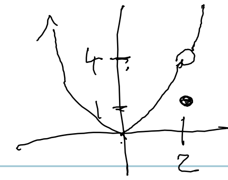
at $x=0$

Here $h(0)=-1$, is defined however

$\lim_{x\to 0^+} h(x) = 1 \neq \lim_{x\to 0^-} h(x) = -1$

So by (2) h is discontinuous at $x=2$

(C) $g(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$ at $x=2$



Here $f(2) = 1$, is defined $\lim_{x \to 2} g(x) = 4$ exists, but

$g(2) = 1 \neq \lim_{x \to 2} g(2)$, so discontinuous

Vertical Asymptotes are also points of discontinuity!

Continuity on a Closed Interval

- A function is continuous on a closed interval $[a,b]$ if

1) It is continuous on the open interval (a,b)
2) It is continuous from the right at a
3) It is continuous from the left at b

Continuous Functions

Polynomial functions: continuous $\forall x \in \mathbb{R}$

Rational functions: $R(x) = \frac{p(x)}{q(x)}$ with p, q polynomials are continuous at all values where $q \neq 0$

Radical Functions: $f(x) = \sqrt[n]{x}$

If n is even, f is continuous at all $x \geq 0$. If n is odd f is continuous everywhere

Exponential Functions: $g(x) = a^x$, $a > 0$ are continuous at all $x \in \mathbb{R}$

Logarithmic Functions: $h(x) = \log_b(x)$ $a > 0$ $a \neq 1$ are continuous $\forall x > 0$

e.g. Find all points of discontinuity for the function

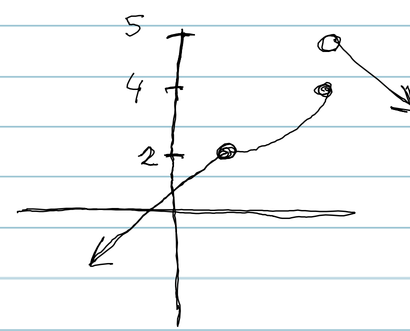
(A) $f(x) = \frac{4x-3}{2x-7}$: Rational $\rightarrow$ $2x-7=0 \Rightarrow x = 7/2$

(B) $g(x) = e^{2x-3}$: exponential $\rightarrow$ No discontinuities

e.g. Find all values of x where the function is continuous

$$f(x) = \begin{cases} x+1 & \text{if } x<1 \\ x^2-3x+4 & \text{if } x \in [1,3] \\ 5-x & \text{if } x>3 \end{cases}$$

- Observe f is continuous at $(-\infty,1)$, $(1,3)$, $(3,\infty)$ at the very least, since it is equal to various polynomial functions at these values.
- However, we must check the places where f is "pasted together" (the endpoints)



$\boxed{x=1}$ $f(1) = 1^2-3+4 = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x+2 = 2$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2-3x+4 = 1^2-3+4 = 2$

So f is indeed continuous at $x=1$

$\boxed{x=3}$ $f(3) = 3^2-3\cdot 3+4 = 9-9+4 = 4$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 5-x = 5-3 = 2$ $\neq$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 3^2-3\cdot3+4 = 9-9+4 = 4$

So ~~that~~ f is NOT continuous at $x=3$

$\Rightarrow$ f is continuous on $\boxed{(-\infty,3)\cup(3,\infty)}$

e.g.) Find the value of k st the function is continuous:

$$g(x) = \begin{cases} \frac{2x^2 - x - 15}{x-3}, & x \neq 3 \\ kx - 1, & x = 3 \end{cases}$$

We need to choose k so that these two separate pieces

$$\begin{matrix} * \\ ** \end{matrix} \begin{cases} \frac{2x^2-x-15}{x-3} \\ kx-1 \end{cases}$$ agree (match up) where they are "pasted" together

First, lets find the limit as $x \to 3$ of $(*)$

$$\lim_{x \to 3} \frac{2x^2-x-15}{x-3} = \frac{(2x+5)\cancel{(x-3)}}{\cancel{(x-3)}} = 2x+5 \quad (x \neq 3)$$

$$= 2 \cdot 3 + 5 = 11$$

Now, when $x=3$

$$kx - 1 = 3k - 1$$

In order for these to agree, we must set them equal and solve for k:

$$2x+5 \Big|_{x=3} = kx-1 \Big|_{x=3}$$

$$\Rightarrow 6+5 = 3k-1$$

$$\Rightarrow 11 = 3k-1$$

$$\Rightarrow k = \frac{12}{3} = \boxed{4}$$

§3.3 : Rates of Change

- One of the main applications of Calc is determining how one variable changes in relation to the other
- For lines, this is easy, the rate of change is the slope.
- We can approximate this for any function $f(x)$

Average Rate of Change

- The average rate of change of a function $f(x)$ with respect to x as x changes from a to b is:

$$f_{avg} = \frac{f(b) - f(a)}{b - a}$$

e.g./ Suppose 89.7% of households had landline phones in 2005, while in 2009 this number reduced to 73.5%. Determine the average rate of change in the percent of landlines in American households per year over 2005 - 2009

- Plug + Chug:

$$f_{avg} = \frac{73.5 - 89.7}{2009 - 2005} = \frac{-16.2}{4} = -4.05$$

Decline (negative) is roughly 4.05% per year.