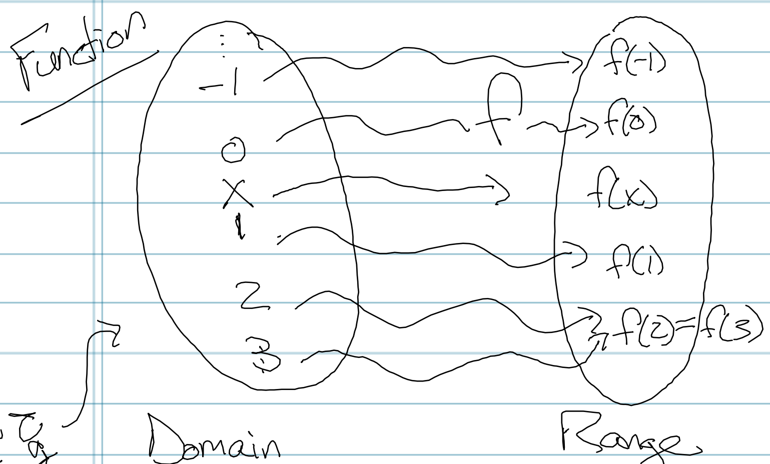
§2.1 : Properties of Functions

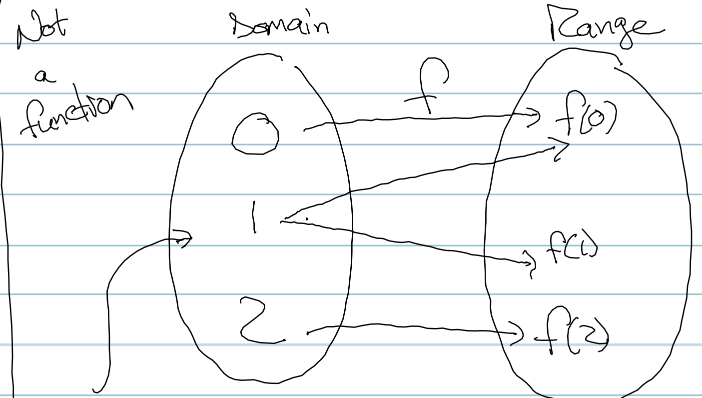
Def^n : A function is a rule that assigns to each element from one set exactly one element from another set

Def^n: The set of all possible values of the independent variables ("the x in f(x)") is called the domain

Def^n: The range is the set of all possible values of the independent variable (All possible f(x), where x is in the domain of f)



Note 2 values in the domain can map to 1 value in the range



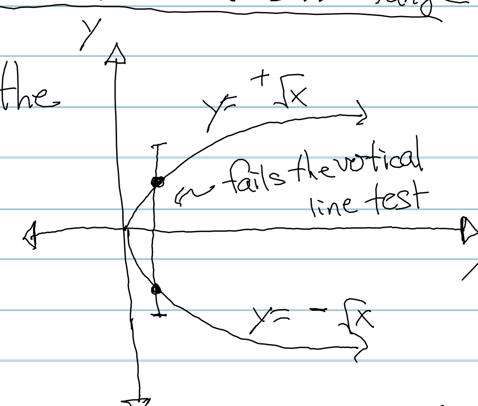
This is NOT a function since 1 value in the domain maps to 2 values in the range

e.g. Is $y^2 = x$ a function?

$y^2 = x \Rightarrow y = \pm\sqrt{x}$

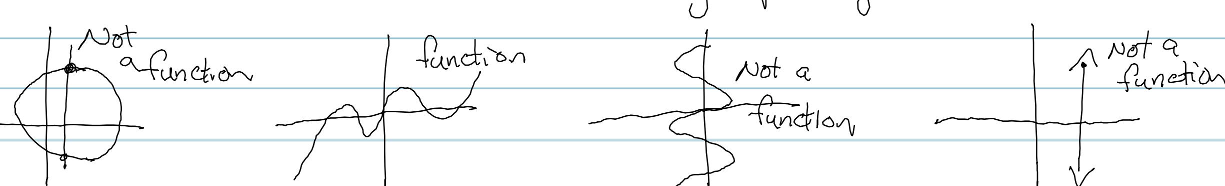
This means that 2 different x-values are mapping to a single y-value. (NOT a function)

Consider the graph



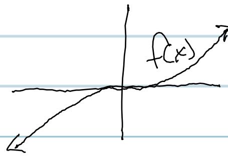
Vertical Line Test: If a vertical line intersects a graph in more than one point, the graph is NOT the graph of a function.

i.e.



e.g.) Find the domain and range of the following functions:

(A) $f(x) = \dfrac{x^3}{x^2+1}$ Since $x^2+1 \neq 0$, $\forall x \in \mathbb{R}$ the domain is $(-\infty, \infty)$



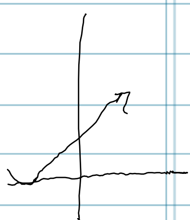
Similarly, since the numerator is growing faster than the denominator, this function ~~lets all~~ will hit every real #, so the range is $(-\infty, \infty)$

(B) $f(x) = \sqrt{x^2+5x+6}$: Here, $f(x)$ is only a real number when descriminant is ≥ 0. Thus we must solve the inequality:

$$x^2+5x+6 = (x+3)(x+2) \geq 0 \Rightarrow \overset{\text{critical pts}}{x=-3,\ x=-2}$$

This leads to the intervals: $(-\infty, -3]$, $[3, -2]$, $[-2, \infty)$

We know at the endpts $f(x)=0$, so the inequality is satisfied



Test Pts:

$x = -4 \Rightarrow 16-20+6 = 2 > 0$ ✓

$x = -2.5 = -\frac{5}{2} \Rightarrow \frac{25}{4} - \frac{25}{2} + 6 = -\frac{25}{2} + \frac{12}{2} = -\frac{13}{2} < 0$ ✗

$x = 0 \Rightarrow 0+0+6 > 0$ ✓

Thus the domain is $\boxed{(-\infty, -3] \cup [-2, \infty)}$

We see that $f(x)$ is never negative, however can $= 0$. Since as $x \to \infty$, $f(x) \to \infty$ the range is: $\boxed{[0, \infty)}$

(C) $f(x) = \dfrac{2}{x^2-9}$ We know the denominator can not equal 0. So set the denominator equal to zero to see what points can not work:

$$x^2-9 = (x+3)(x-3) = 0 \Rightarrow x = \pm 3$$

$$\Rightarrow \boxed{\text{Domain} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)}$$

We see, since the numerator can not equal 0, $f(x) \neq 0$, $\forall x$. Note $f(x)$ is arbitrarily large when x is near ± 3. Thus

$$\boxed{\text{Range} = (-\infty, 0) \cup (0, \infty)}$$

Def^{n}: A function is called even if $f(-x) = f(x)$

i.e. it is symmetric over the y-axis

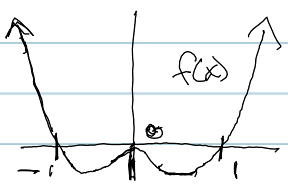
A function is called odd if $f(-x) = -f(x)$

i.e. symmetric about the origin

e.g./ Determine if function is even, odd, or neither

(A) $f(x) = x^4 - x^2$: $\Rightarrow f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x)$

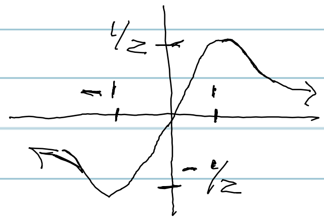
So even, which can be seen by the graph



Note: $x^4 - x^2 = x^2(x^2-1) = x^2(x+1)(x-1)$

(B) $f(x) = \dfrac{x}{x^2+1} \Rightarrow f(-x) = \dfrac{-x}{(-x)^2+1} = \dfrac{-x}{x^2+1} = -f(x)$

So odd, which can also be seen by graph

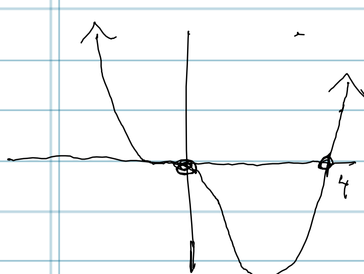


(C) $f(x) = x^4 - 4x^3 \Rightarrow f(-x) = (-x)^4 - 4(-x)^3$

$= x^4 - 4(-x)^3 = x^4 + 4x^3$

$\neq f(x)$

$\neq -f(x)$

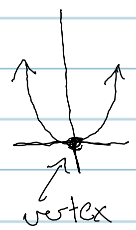


So, Neither

$x^4 - 4x^3 = x^3(x-4)$

§ 2.2: Quadratic Functions (Translation/Reflection)

Recall a quadratic function is of the form $\boxed{f(x)=ax^2+bx+c}$ where $a,b,c \in \mathbb{R}$, $a \neq 0$



The graphs of quadratic equations are called parabolas. The vertex of a parabola is the point on the parabola where the slope of the tangent line is equal to 0. and can be found via an equation:

Recall the quadratic ~~equation~~ formula: $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

That is, x is the solution to the quadratic equation $ax^2+bx+c=0$. Now, we have the explicit formula for both roots of this equation. Since quadratic's are symmetric about the vertex, we know the vertex is halfway between both roots. Thus:

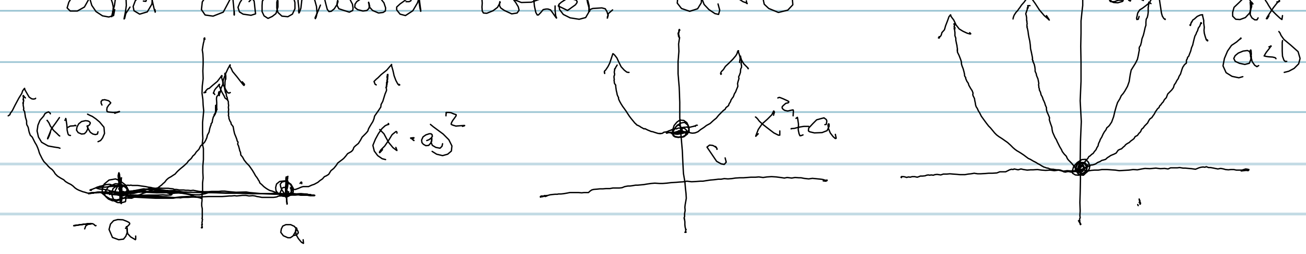
$$x_0 = \frac{\frac{-b}{2a} + \cancel{\frac{\sqrt{b^2-4ac}}{2a}} + \frac{-b}{2a} - \cancel{\frac{\sqrt{b^2-4ac}}{2a}}}{2} = \frac{-2b}{2a}\cdot\frac{1}{2} = \boxed{\frac{-b}{2a}}$$

Thus the vertex of a quadratic equation is given as:

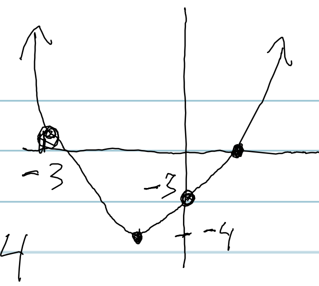
$$\boxed{(x_0, y_0) = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)}$$

when the eqn is of the form $f(x)=ax^2+bx+c$

* Note the graph opens upward when $a>0$ and downward when $a<0$



e.g.) Graphing Quadratic Function: $f(x) = x^2 + 2x - 3$



$a = 1,\ b = +2,\ c = -3$

vertex: $x_0 = \frac{-b}{2a} = \frac{-2}{2} = -1$

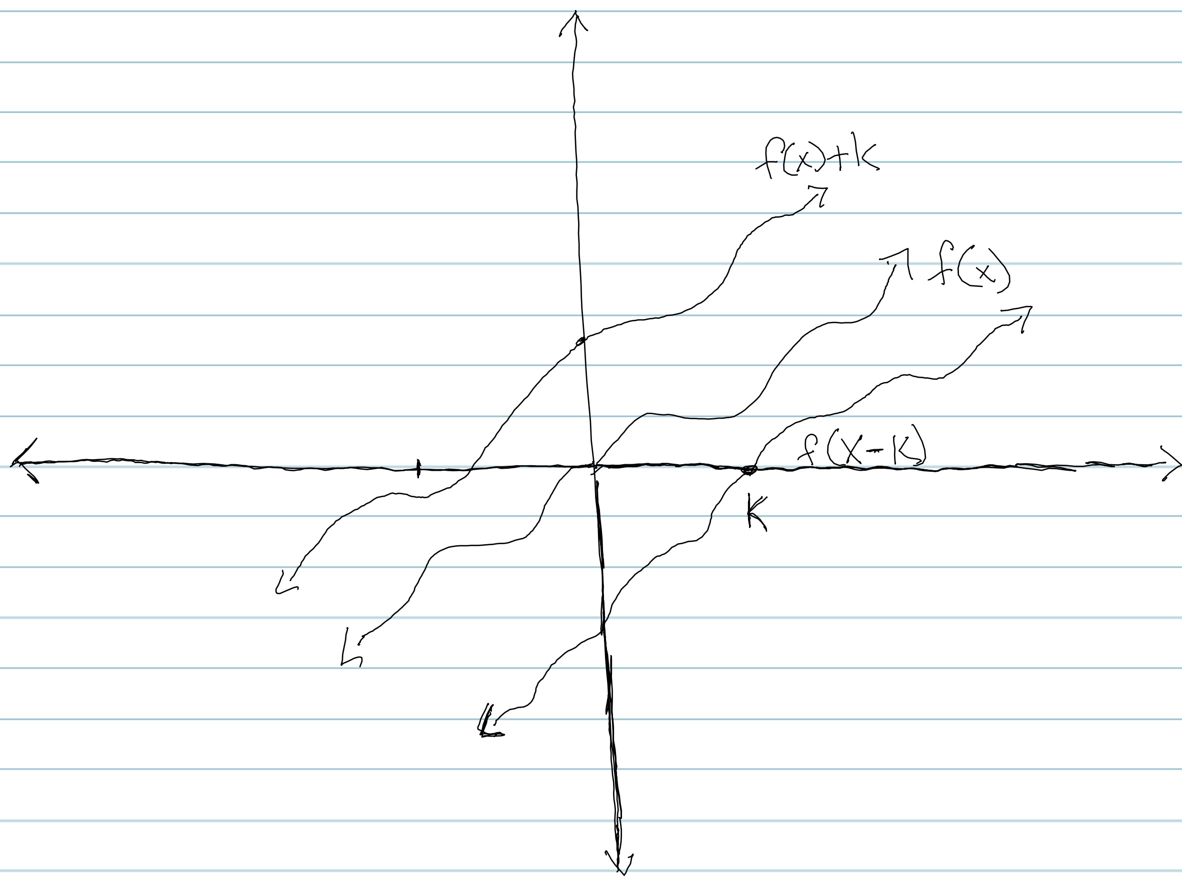
$y_0 = f(x_0) = (-1)^2 + (2(-1)) - 3 = 1 - 2 - 3 = -4$

Zeros: Factor: $x^2 + 2x - 3 = 0 = (x+3)(x-1) \Rightarrow x = 1, -3$

y-int: $f(0) = 0 + 0 - 3 = -3$

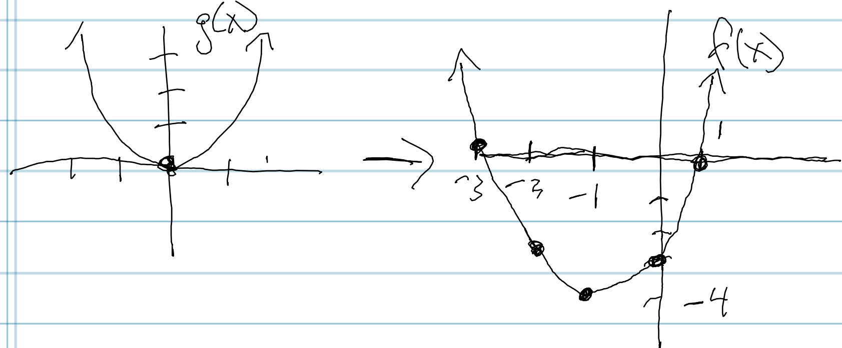
Translations & Reflections: Let f be any function, $K > 0$

- the graph of $f(x) + K$ is the graph of $y = f(x)$ translated upward by K units
- the graph of $f(x) - K$ is $y = f(x)$ shifted downward K units
- the graph of $f(x \pm K)$ is that of $f(x)$ shifted left/right K units
- the graph of $-f(x)$ is that of $f(x)$ reflected over x-axis
- the graph of $f(-x)$ is that of $f(x)$ reflected across y-axis



e.g. Graph $f(x)=(x+1)^2-4$ using translations, reflections of $g(x)=x^2$

Note we can rewrite f as $f(x)=g(x+1)-4$

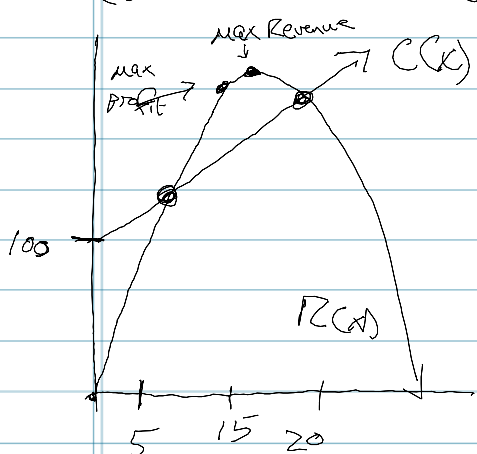


x	$g(x)$
-2	4
-1	1
0	0
1	1
2	4

x	$f(x)$
-3	0
-2	-3
-1	-4
0	-3
1	0

e.g. Deli owner finds that revenue from producing x pounds of vegetable cream cheese is given by $R(x)=-x^2+30x$ while cost is given by $C(x)=5x+100$

(A) Find minimum break even



As usual, find the intersection(s)

$$C(x)=R(x)$$

$$\Leftrightarrow 5x+100=-x^2+30x$$

$$\Leftrightarrow x^2-25x+100=0=(x-5)(x-20)$$

$$\Rightarrow x=5 \text{ or } x=20$$

We see the minimum break even pt is at $x=5$

$$\Rightarrow C(5)=5(5)+100=125$$

(B) Find Maximum Revenue: We know this is a quadratic opening downwards so max is at vertex:

$R(x)=-x^2+30x \Rightarrow a=-1, b=30$

$$\Rightarrow x=\frac{-b}{2a}=\frac{30}{2}=15 \text{ pounds} ; \quad R(15)=-(15)^2+30(15)=\$225$$

(C) Find Max Profit: $P(x)=R(x)-C(x)=-(5x+100)+(-x^2+30x)$

$=-x^2+25x-100$

The max is at vertex

$$\Rightarrow x=\frac{-b}{2a}=\frac{-25}{-2}=\boxed{12.5 \text{ lbs}}, \quad P(12.5)=-(12.5)^2+25(12.5)-100=$$

§2.3: Polynomial and Rational Functions

Def^n: A polynomial of degree n is of the form: $f(x) = a_0 + a_1 x + \dots + a_n x^n$
with $a_i \in \mathbb{R}$ (real coefficients) and $a_n \neq 0$ where a_n is the "leading coefficient"

Properties of Polynomials

(1.) A polynomial function of degree n can have at most $(n-1)$ turning pts.
Conversely, a polynomial function w/ n turning pts must be atleast degree $n+1$

(2.) The graph of an even degree polynomial has both tails going up or down
If the degree is odd the ends go in opposite directions

(3.) If the graph goes "up" as x becomes large and positive, the the leading coefficient is positive. The opposite (graph goes down when $x \to +\infty$) is true if leading coefficient is negative.

Rational Functions

Rational function defined by: $f(x) = \frac{p(x)}{q(x)}$
where $p(x)$, $q(x)$ are polynomial functions with $q(x) \neq 0$
(when graphing, plot pts on a table and use assymtotes)

Vertical Assymtotes: The vertical assymptotes of the rational function $f(x) = \frac{p(x)}{q(x)}$ occur at the zeros of $q(x)$
(i.e. if $q(k) = 0$, then $x = k$ is a vertical assymtote)

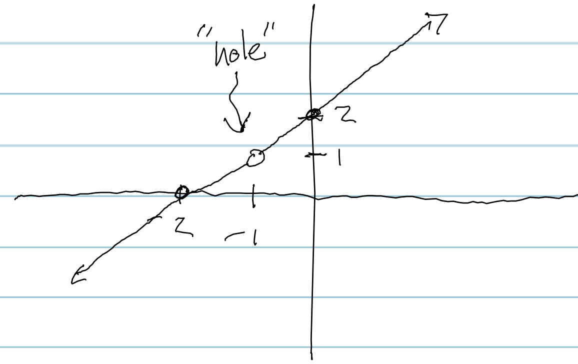
Here $\deg(p(x)) \equiv$ "degree of $p(x)$"

Horizontal Assymtotes: Given rational function $f(x) = p(x)/q(x)$
(1) If $\deg(q(x)) > \deg(p(x))$ then there is a horizontal assymtote at $y = 0$
(2) If $\deg(q(x)) = \deg(p(x))$, then horizontal assymtote occurs at quotient of leading coefficients
(3) If $\deg(p(x)) > \deg(q(x))$, then f has NO horizontal assymtotes

Holes in graphs: If $f(x) = \frac{p(x)}{q(x)} = \frac{r(x)(x-k)}{s(x)(x-k)}$, with p, q, r, s polynomial functions then the graph of f is that of $\frac{r(x)}{s(x)}$ with the point $\left(k, \frac{r(k)}{s(k)}\right)$, deleted. This point is called a "hole" in the graph of f

e.g.) Graph: $f(x) = \frac{x^2+3x+2}{x+1} = \frac{(x+1)(x+2)}{(x+1)} = x+2$

This is simply the graph of $x+2$ with a hole at $(-1, 1)$



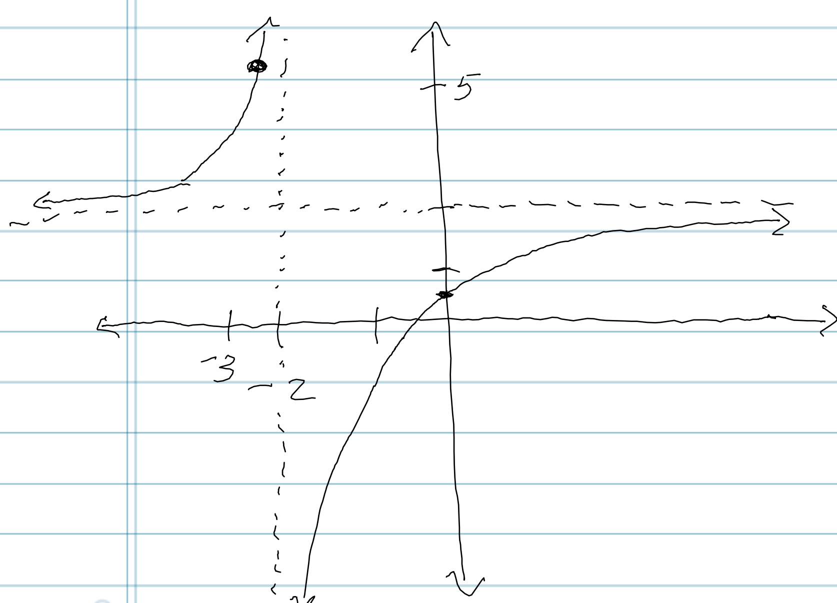
e.g.) Graph $f(x) = \frac{4x+2}{2x+4}$

Here we have $\deg(4x+2) = \deg(2\cancel{x}+4) = 1 \Rightarrow$ Horizontal assymptote at: $y = \frac{4}{2} = 2$ (leading coefficient!)

Also, we have a zero in the denominator:

$2x+4 = 0 \Rightarrow x = -2$, hence we have a vertical assymtote at this pt. Other pts:

x	$f(x)$
0	$\frac{2}{4} = \frac{1}{2}$
-3	$\frac{-10}{-2} = 5$

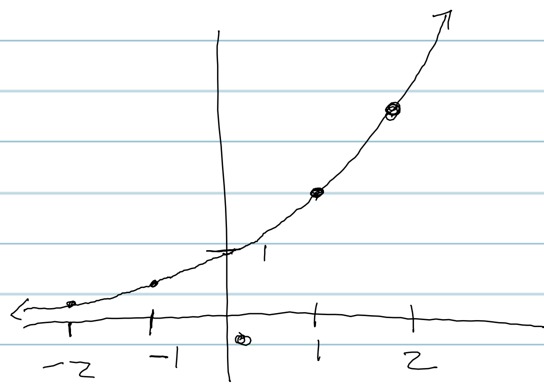


§2.4 : Exponential Functions

An exponential function with base, a, is defined as $\boxed{f(x) = a^x}$; $a > 0$ and $a \neq 1$

e.g.) Graph $y = 2^x$

x	y
-2	$\frac{1}{4}$
-1	$1/2$
0	1
1	2
2	4



Exponential Eqn's

Base exponent Property: If $a > 0$, $a \neq 1$ then:

$$a^x = a^y \Rightarrow x = y$$

e.g.) (A) Solve $9^x = 27$. First rewrite in terms of the same base (if possible)

$$\Rightarrow (3^2)^x = 3^3$$

$$\Rightarrow 3^{2x} = 3^3 \Rightarrow 2x = 3 \Rightarrow \boxed{x = \frac{2}{3}}$$

(B) $32^{2x-1} = 128^{x+3}$

$$\Rightarrow (2^5)^{2x-1} = (2^7)^{x+3}$$ Same base, so set exponents equal

$$\Rightarrow 5(2x-1) = 7(x+3)$$

$$\Rightarrow 10x - 5 = 7x + 21 \Rightarrow 3x = 26 \Rightarrow \boxed{x = \frac{26}{3}}$$

Jim Compound Interest

Def^n: interest: cost of borrowing money or return on investment
principal: Am$^{\underline{t}}$ borrowed or invested
rate of interest: percent per year earned/owned (aka APR)

Simple Interest (only initial investment/loan accrues interest

$$\boxed{I = Prt}$$; interest is calculated by multiplying original principal (P) at rate (r) for time (t)

With compound interest, the interest gained/paid per time period is incorporated in finding interest paid/gained in next time step:

Compound Interest: If P dollars are invested at a yearly rate of r·100 % per year and compounded m time. per year, the compounded amount is given by:

$$\boxed{A = P\left(1+\frac{r}{m}\right)^{tm}}$$

e.g.) Determine the compounded amount on a principal of \$9000 at 6% annual interest compounded semiannali for 4 years:
2 times per yr

$A = ?$; $P = \$9000$; $r = .06$, $t = 4$, $m = 2$

$$\Rightarrow A = 9000\left(1+\frac{.06}{2}\right)^{2\cdot 4} = 9000(1.03)^8$$ in 4 yrs were compound. 8 times

$$\approx \boxed{\$11,400.93}$$

So this is your new balance in 4 years at this rate.

The number e: As m gets larger and larger the value of $\left(1+\frac{1}{m}\right)^m$ becomes arbitrarily close to the # whose approximate value is 2.718281829... :

$$e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$$

Notice we can write our compound interest formula as:

$$A = P\left(1+\frac{r}{m}\right)^{mt} = P\left[\left(1+\frac{1}{\frac{m}{r}}\right)^{m/r}\right]^{rt}$$

Now let $\frac{m}{r}\to\infty$ (i.e. the # of times we are compounding is getting very large)

$$\Rightarrow \lim_{m/r\to\infty} A = \lim_{m/r\to\infty} P\left[\left(1+\frac{1}{m/r}\right)^{m/r}\right]^{rt} = Pe^{rt}$$

So as our compounding period gets ∞'ly small (so $\frac{m}{r}$ gets large) we find:

Continuous Compounding Interest

If a deposit of P dollars is invested at an interest rate of $r\cdot 100\%$ per year continuously $(m\to\infty)$ for t years the amount earned is:

$$\boxed{A = Pe^{rt} \text{ dollars}}$$

§2.5: Logarithmic Functions

- How do we undo an exponential function?

__Def^n__: For $a>0$, $a\neq 1$, $x>0$ we take

$$\boxed{\begin{array}{l} y=\log_a x \\ a^y = x \end{array}}$$

$y=\log_a x$ to be the real # for which $\Rightarrow$ $a^y=x$ is true

__Def^n__: If $a>0$, $a\neq 1$ then the logarithmic function of base a is defined to be

$$f(x) = \log_a x \quad \text{for } x>0.$$

__Properties of Logarithms__

Let x,y positive real #'s, r be any real #, $a>0$, and $a\neq 1$

$(\equiv x,y \in \mathbb{R}_{>0}\ ;\ r\in\mathbb{R},\ a\in \mathbb{R}_{>0} / \{1\})$

(A) $\log_a xy = \log_a x + \log_a y$ $\Leftarrow (a^x a^y = a^{x+y})$ __Product Rule__

(B) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ __Quotient rule__

(C) $\log_a x^r = r\log_a x$ __Power Rule__

(D) $\log_a a = 1$ $\quad (a^1 = a)$

(E) $\log_a 1 = 0$ $\quad (a^0 = 1)$

* Note we denote $\ln(x) = \log_e x$ and when No base is indicated, assume base 10 $(\log x = \log_{10} x)$

* ~~$\log_e$~~ $\ln(x) = \log_e(x)$ is called the natural logarithm

Property of Logarithmic Equality: For any logarithm base a and expressions x, y we know

$$\log_a x = \log_a y \Rightarrow x = y$$

$a^x = a^y$

e.g. Simplify:

(A) $\log_4 64 = \log_4 4^3 = 3(\cancel{\log_4 4}^{1}) = 3$

(B) $\log_5 80 = \log_5(5 \cdot 16) = \log_5 5 + \log_5 16 = \boxed{1 + \log_5 16}$

(C) $\log_2(1/16) = \log_2 2^{-4} = -4\log_2 2 = \boxed{-4}$

(D) $\log_{1/5} 25 = \log_{1/5}(5^2) = \log_{1/5}\left(\frac{1}{5}\right)^{-2} = -2\cancel{\log_{1/5}(1/5)}$
$= \boxed{-2}$

Change of Base Formula

If x is any positive # and a, b are positive with $a, b \neq 1$ then:

$$\log_a x = \boxed{\frac{\log_b x}{\log_b a}}$$

e.g. (A) $\log_4 x = \frac{3}{2} \longrightarrow x = 4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$

(just think of taking base 4 of both sides)

$\cancel{4}^{\cancel{\log_4} x} = 4^{3/2} \Rightarrow \boxed{x = 4^{3/2}}$

(B) $\log_2 x - \log_2(x-1) = 1 \Rightarrow \cancel{2^{\log_2}}\left(\frac{x}{x-1}\right) = 2^1$

$\Rightarrow \left(\frac{x}{x-1}\right) = 2^1 \Rightarrow x = 2x - 2 \Rightarrow x = 2$

(C) $\log x + \log(x-3) = 1 \Rightarrow 10^{\log(x^2-3x)} = 10^1$

$\Rightarrow x^2-3x=10 \Rightarrow x^2-3x-10=0$

$(x-5)(x+2)$

$\Rightarrow \boxed{x=5 \text{ or } \cancel{x=-2}}$ Wait! No Negatives in logarithms! Throw out $x=-2$!!

(D) $3^{2x} = 4^{x+1}$

$\Rightarrow \ln 3^{2x} = \ln 4^{(x+1)} \Rightarrow 2x\ln 3 = (x+1)\ln 4$

$\Rightarrow (2\cdot \ln 3)x = (\ln 4)x + \ln 4$

$\Rightarrow (\ln 9 - \ln 4)x = \ln 4 \Rightarrow \ln\left(\frac{9}{4}\right)x = \ln 4$

$\Rightarrow x = \dfrac{\ln 4}{\ln\left(\frac{9}{4}\right)} \approx 1.710$

(E) $5e^{2x} = 9 \Rightarrow e^{2x} = \frac{9}{5} \Rightarrow \ln e^{2x} = \ln\left(\frac{9}{5}\right)$

$\Rightarrow 2x \ln e = \ln\frac{9}{5} \Rightarrow \boxed{x = \ln{}^{9}\!/\!_{5}\,/\,2}$

Change of Exponential Base Formula

For every positive real # a $(a \in \mathbb{R}_{>0})$

$\boxed{a^x = e^{(\ln a)x}}$ (since $e^{\ln a} = a$) ← in case you want to use base e over base

In particular if $b>0$, $b \neq 1$ then

$\boxed{a^x = b^{(\log_b a)x}}$

§2.6: Applications - Growth, Decay, Finance

Exponential Growth and Decay: Let y_0 be the amount or quantity present at time $t=0$. The quantity is said to grow or decay exponentially if for some constant, k, the amount present at time t is given by:

$$y = y_0 e^{kt} \quad \text{where } \begin{array}{l} k>0 \Rightarrow \text{growth} \\ k<0 \Rightarrow \text{decay} \end{array}$$

e.g.) Carbon Dating

(A) The isotope Carbon-14 (C_{14}) decays with the constant $k = \left(-\frac{\ln 2}{5600}\right)$. Find the half life of C_{14} in years.

- Here, we want time, t where amount present (y) is half of what we originally had: $y = y_0/2$. Plug this into LHS and solve for t:

$$y = \frac{\cancel{y_0}}{2} = y_0 e^{kt} = \cancel{y_0} e^{-\left(\frac{\ln 2}{5600}\right)t}$$

$$\Rightarrow \frac{1}{2} = e^{-\left(\frac{\ln 2}{5600}\right)t} \Rightarrow \ln\left(\frac{1}{2}\right) = \cancel{\ln} e^{-\left(\frac{\ln 2}{5600}\right)t}$$

$$\Rightarrow \ln\frac{1}{2} = \frac{-\ln 2}{5600}t \Rightarrow \ln(2)^{-1} = \frac{-\ln 2}{5600}t$$

$$\Rightarrow -\cancel{\ln 2} = \frac{-\cancel{\ln 2}}{5600}t \Rightarrow \boxed{t = 5600 \text{ yrs}}$$

(B) Suppose some charcoal was found at an archaeological dig and had 1/4 the quantity of C_{14} a living sample of wood ordinarily has. Estimate the age of the charcoal.

(Note: if y_0 is the original quantity of C_{14}, there is now $\frac{y_0}{4}$ left)

$$\frac{1}{4}\cancel{y_0} = \cancel{y_0} e^{\frac{-\ln 2}{5600}t} \Rightarrow \ln\frac{1}{4} = -2\cancel{\ln 2} = \frac{-\cancel{\ln 2}}{5600}t$$

$$\Rightarrow t = 5600 \cdot 2 = \boxed{11{,}200 \text{ yrs old}}$$

Def^{n}: The percent owned/~~earned~~ earned on a loan/investment per yr is known as the <u>effective rate</u>

<u>Effective Rate For Compound Interest</u>

If r is the annual rate of interest and m is the # of compounding periods per yr. Then the effective rate, r_E is:

$$\boxed{r_E = \left(1+\frac{r}{m}\right)^m - 1}$$

<u>Effective rate for Continuous Compounding</u>

If interest is compounded annually at annual rate, r, then the effective rate, r_E, is given by: $\boxed{r_E = e^r - 1}$

e.g.) Find the effective rate corresponding to each stated annual rate, r

(A) $r = 6\%$ compounded quarterly: $r = .06$, $m = 4$

$$\Rightarrow r_E = \left(1+\frac{.06}{4}\right)^4 - 1 = (1.015)^4 - 1 \approx .0614 = \boxed{6.14\%}$$

(B) 6% compounded continuously:

$$\Rightarrow r_E = e^{.06} - 1 \approx .0618 \text{ or } \boxed{6.18\%}$$

e.g.) If you invest \$25,000 at 7.2% APR compounded quarterly how long will it take to have a total \$40,000 accrued.

$P = 25{,}000$, $r = .072$, $m = 4$. Plug into compound interest formula:

$$A = P\left(1+\frac{r}{m}\right)^{mt} \quad ; \quad t = ?$$

$$\Rightarrow 40000 = 25{,}000\left(1+\frac{.072}{4}\right)^{4t} = 25{,}000(1.018)^{4t}$$

$$\Rightarrow \frac{40}{25} = (1.018)^{4t} \Rightarrow \ln\left(\frac{40}{25}\right) = 4t \cdot \ln(1.018)$$

$$\Rightarrow t = \frac{\ln(40/25)}{4\ln(1.018)} \approx \boxed{6.586 \text{ yrs}}$$