Math 255 – Spring 2022
Continued fractions

Definition 1. A **simple continued fraction** is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}}}},$$

where $a_0 \in \mathbb{Z}$ and for $i \geq 1$, $a_i \in \mathbb{Z}$ and $a_i \geq 1$. For compactness of notation, we usually write $[a_0; a_1, a_2, a_3, \ldots, a_n, \ldots]$ for this expression.

Further, if α is a real number and $\alpha = [a_0; a_1, a_2, a_3, \ldots, a_n, \ldots]$, then we say that $[a_0; a_1, a_2, a_3, \ldots, a_n, \ldots]$ is the **continued fraction expansion** of α.

If $\alpha \in \mathbb{Q}$, then α has exactly two distinct continued fraction expansions, and both of them are finite. If $\alpha \in \mathbb{R}$ but $\alpha \notin \mathbb{Q}$, then α has a unique continued fraction expansion, which is infinite. We now show how to compute the continued fraction expansion of a number:

Example 2. Let $\alpha = \frac{47}{17}$, and suppose that we want to express α as a simple continued fraction. The first number to find is a_0. Since $\frac{47}{17} = 2 + \frac{13}{17}$, we have that $a_0 = 2$.

Now we compute a_1. If we notice the pattern of the continued fraction, we need to write $\frac{47}{17} = 2 + \frac{1}{??}$. So what we will do is write

$$\frac{47}{17} = 2 + \frac{1}{\frac{17}{13}}.$$

Now since $\frac{17}{13} = 1 + \frac{4}{13}$, we have

$$\frac{47}{17} = 2 + \frac{1}{1 + \frac{4}{13}},$$

and $a_1 = 1$. We can keep going like this, and we do, until we're done!

$$\frac{47}{17} = 2 + \cfrac{1}{1 + \cfrac{1}{\frac{13}{4}}} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{4}}}.$$

So we have $a_2 = 3$. But now we see that the last fraction already has a numerator of 1, so we already have $\frac{47}{17}$ in the correct form, and we are done:

$$\frac{47}{17} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4}}},$$

so $a_0 = 2, a_1 = 1, a_2 = 3$, and $a_4 = 4$. The continued fraction expansion is finite.

There are many fun things to say about continued fractions, but for this week we'll stick to some concepts that have actual practical applications.

One important quantity will be the so-called **convergents** of a continued fraction:

Definition 3. Given a continued fraction $[\,a_0; a_1, a_2, a_3, \ldots, a_n, \ldots]$, its **$k$th convergent** is the number given by

$$\frac{p_k}{q_k} = [\,a_0; a_1, a_2, a_3, \ldots, a_k\,].$$

As one might expect, when $\frac{p_k}{q_k}$ is written in lowest terms, the number p_k is called the **numerator of the kth convergent** of $[\,a_0; a_1, a_2, a_3, \ldots, a_n, \ldots]$ and q_k is called the **denominator of the kth convergent** of $[\,a_0; a_1, a_2, a_3, \ldots, a_n, \ldots]$.

With this notation in place, we have

Proposition 4. The numerator p_k and the denominator q_k of the kth convergent of $[\,a_0; a_1, a_2, a_3, \ldots, a_n, \ldots]$ are given recursively by the following formulae:

$$\begin{aligned} p_{-2} &= 0, \quad p_{-1} = 1 \\ q_{-2} &= 1, \quad q_{-1} = 0, \end{aligned}$$

and for $k \geq 0$,

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2}, \\ q_k &= a_k q_{k-1} + q_{k-2}. \end{aligned} \tag{1}$$

Remark 5. Note that in the definition, the fraction $\frac{p_k}{q_k}$ is **in lowest terms**. Accordingly, the proposition gives $\frac{p_k}{q_k}$ *also* in lowest terms! This can help you spot a mistake in your calculations if you get any factors in common between p_k and q_k for some k.

One application of convergents and continued fraction expansions is the following:

Theorem 6. Let α be any real number. If the rational number $\frac{a}{b}$, where $b \geq 1$ and $\gcd(a, b) = 1$ satisfies

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2},$$

then $\frac{a}{b}$ is one of the convergents $\frac{p_k}{q_k}$ of the continued fraction expansion of α.

As a kind of converse to this theorem, one might wonder if its convergents ever get this close to α, and they do:

Proposition 7. Let α be an irrational number. Then for any two consecutive convergents, at least one of them, which we will denote $\frac{p}{q}$, satisfies the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$