Name:

Problem 1: *Prove that* $\mathbb{Q}(\sqrt[3]{2})$ *is not a subfield of any cyclotomic field over* $\mathbb{Q}$.

As a hint, you may assume that the Galois closure K *of* $\mathbb{Q}(\sqrt[3]{2})$ *over* $\mathbb{Q}$ *has* $\mathrm{Gal}(K/\mathbb{Q}) \cong D_3$.

Solution: Suppose for a contradiction that $\mathbb{Q}(\sqrt[3]{2})$ is indeed contained in some cyclotomic field $\mathbb{Q}(\zeta_n)$. Since $\mathbb{Q}(\zeta_n)$ is Galois over $\mathbb{Q}$, it follows by the definition of Galois closure that K, the Galois closure of $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$, is contained in $\mathbb{Q}(\zeta_n)$. By the Fundamental Theorem of Galois Theory, this means that $\mathrm{Gal}(K/\mathbb{Q})$ is a quotient of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$. But $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is abelian and all quotients of abelian groups are abelian, which is a contradiction since D_3 is not abelian.