Name:

Problem 1: *Prove that the map*

$$\begin{aligned}\sigma\colon \mathbb{Q}(\sqrt{3}) &\to \mathbb{Q}(\sqrt{3})\\ a+\sqrt{3}b &\mapsto a-\sqrt{3}b\end{aligned}$$

is a field isomorphism.

Solution: A field isomorphism is a bijective ring homomorphism between two fields. We first show that it is a ring homomorphism: if $a+b\sqrt{3}$ and $c+d\sqrt{3}$ are two elements of $\mathbb{Q}(\sqrt{3})$, then we have

$$\begin{aligned}\sigma((a+b\sqrt{3})+(c+d\sqrt{3})) &= \sigma((a+c)+\sqrt{3}(b+d))\\ &= (a+c)-\sqrt{3}(b+d)\\ &= (a-\sqrt{3}b)+(c-\sqrt{3}d)\\ &= \sigma(a+\sqrt{3}b)+\sigma(c+\sqrt{3}d),\end{aligned}$$

and

$$\begin{aligned}\sigma((a+b\sqrt{3})\cdot(c+d\sqrt{3})) &= \sigma((ac+3bd)+\sqrt{3}(ad+bc))\\ &= (ac+3bd)-\sqrt{3}(ad+bc)\\ &= (a-\sqrt{3}b)\cdot(c-\sqrt{3}d)\\ &= \sigma(a+\sqrt{3}b)\cdot\sigma(c+\sqrt{3}d).\end{aligned}$$

We now show that it is bijective. First, since it is a map of fields, it is either the zero map or injective. Since it is not the zero map ($\sigma(\sqrt{3})=-\sqrt{3}\neq 0$ for example), it is injective. It is also surjective: Given any $a+b\sqrt{3}\in\mathbb{Q}(\sqrt{3})$, we have that $\sigma(a-b\sqrt{3})=a+b\sqrt{3}$.