Math 395 - Spring 2020
Homework 8

This homework is due on Monday, March 16.

All of these problems must be typed up.

1. Let R be an integral domain and assume R contains a subring F that is a field (R and F have the same 1). Prove that if R is finite dimensional as a vector space over F then R is a field.

2. Let x and y be independent indeterminates over the field $\mathbb{C}$ of complex numbers, and let $R = \mathbb{C}[x, y]/(x^2 - y, y^2 - x)$.

   (a) Explain why R is a finite dimensional vector space over $\mathbb{C}$, and find its dimension.

   (b) Prove that R is isomorphic to $\mathbb{C}[x]/(x^4 - x)$.

   (c) Show that R is (ring) isomorphic to the direct product of four copies of $\mathbb{C}$.

3. Let R be the following quotient ring of the polynomial ring with rational coefficients:

$$R = \mathbb{Q}[x]/(x^6 - 1).$$

   (a) Find all ideals of R. (Be sure to justify that you found them all.)

   (b) Determine which of the ideals of (a) are maximal, and for each maximal ideal M describe the quotient ring R/M.

   (c) Exhibit an explicit (nonzero) zero divisor in R.

   (d) Does R contain any nonzero nilpotent elements? (Briefly justify.)

4. Let $R = \mathbb{R}[x]/(x^4 - 1)$, so R is a commutative ring with 1.

   (a) Show that all ideals of R are principal.

   (b) Find a generator for each maximal ideal of R.

   (c) For each maximal ideal $\mathfrak{m}$, describe an isomorphism from $R/\mathfrak{m}$ to either $\mathbb{R}$ or $\mathbb{C}$.

5. Let R be the ring of all continuous real valued functions on the closed interval $[0, 1]$ (under the usual pointwise addition and multiplication of functions). Let

$$M = \{f \in R \mid f(1/2) = 0\}.$$

   (a) Prove that M is a prime ideal and identify the quotient ring (as a well-known ring).

   (b) Prove that M is not a principal ideal.

(c) Exhibit an infinite properly increasing chain of ideals of R:

$$I_1 \subset I_2 \subset I_3 \subset \cdots \quad \text{and let} \quad I = \bigcup_{i=1}^{\infty} I_i$$

(where you need not reprove that I is an ideal). Explain why I could not be finitely generated. (Hint: One way is to consider ideals of functions that vanish on certain sets.)

6. Let R be the ring of all *continuous* real valued functions on the closed interval $[0,1]$. For each $a \in [0,1]$, let $M_a = \{f \in R \mid f(a) = 0\}$.

   (a) Find all units in R.

   (b) Give an explicit example of a nonzero zero divisor in R.

   (c) Prove that M_a is a maximal ideal in R.

   (d) Prove that there is a countable subset $\{a_1, a_2, a_3, \ldots\}$ of $[0,1]$ such that $\cap_{i=1}^{\infty} M_{a_i} = 0$.