Math 395 - Spring 2020
Final Exam

The Final Exam will be graded as follows:

10/10 for six complete problems

9.5/10 for four complete problems and substantial progress on the other two problems

8.5/10 for nine complete lettered parts

6/10 for six complete lettered parts

3/10 for three complete lettered parts

Section B: Ring Theory

1. Let $F_1, F_2, \ldots, F_n$ be fields.
   (a) Explicitly describe all the ideals in the direct product ring $F_1 \times F_2 \times \cdots \times F_n$. (Explain briefly why your list is complete.)
   (b) Which of the ideals in (a) are prime? (Justify.)
   (c) Which of the ideals in (a) are maximal? (Justify.)

2. Let R be a Principal Ideal Domain, let p and q be distinct primes in R, and let $a = p^\alpha q^\beta$ for some $\alpha, \beta \in \mathbb{Z}^+$. Let M be any R-module annihilated by (a). Prove that

$$M \cong M_p \oplus M_q,$$

   where M_p is the submodule of M annihilated by (p^α) and M_q is the submodule of M annihilated by (q^β). Do not quote a theorem for this; please provide a direct proof.

3. (a) Find all possible rational canonical forms for a 4×4 matrix over $\mathbb{Q}$ that satisfies $A^6 = I$ (where I is the identity matrix).
   (b) Find all possible rational canonical forms for a 4×4 matrix A over $\mathbb{F}_2$ that satisfies $A^6 = I$.

Section C: Field Theory

4. (a) Compute the Galois group of the splitting field of the polynomial $x^3 - 2$ over $\mathbb{Q}$.
   (b) Compute the Galois group of the splitting field of the polynomial $x^7 - 2$ over $\mathbb{Q}$. (Hint: This computation is similar to the one in part (a).)

5. Let K be the splitting field of $x^{61} - 1$ over the finite field $\mathbb{F}_{11}$.
   (a) Find the degree of K over $\mathbb{F}_{11}$.
   (b) Draw the lattice of all subfields of K. (You need not give generators for these subfields.)
   (c) How many elements $\alpha \in K$ generate the multiplicative group $K^\times$?
   (d) How many primitive elements are there for the extension $K/\mathbb{F}_{11}$? (In other words, how many β are there such that $K = \mathbb{F}_{11}(\beta)$?)

6. Let V be the field with 3^6 elements and let $F \subset V$ be the field with 3 elements, so that V is a 6-dimensional vector space over F. Define

$$T\colon V \to V \quad \text{by} \quad T(a) = a^3 \quad \text{for all } a \in V.$$

   (T is called the *Frobenius automorphism of* V.)
   (a) Show that T is an F-linear transformation from V to itself, and that $T^6 = I$, where I is the identity linear transformation. (You may quote without proof facts about finite fields and their Galois theory as long as you state these explicitly.)
   (b) Show that $x^6 - 1$ is both the minimal polynomial and the characteristic polynomial of the linear transformation T. (Hint: Suppose T satisfies a polynomial of lower degree and derive a contradiction.)
   (c) Find the Jordan canonical form of the linear transformation T.