Name:

Problem 1: *Pat and Sam are studying for their final exam in Topology.*

Sam presents the following theorem and proof to Pat: "Theorem: The preimage of a compact set by a continuous map is compact.

Proof: Let $f\colon X \to Y$ be a continuous map, with Y compact. We show that $X = f^{-1}(Y)$ is compact. Let $\mathcal{A}$ be a cover of Y by open sets. Since Y is compact, there is a finite subcover. Now the inverse images of these open sets are open and cover X, and therefore X is compact."

*Pat says: "This doesn't feel right, I thought that the **image** of a compact set by a continuous map was compact..."*

Who is correct? Pat or Sam? Justify your answer briefly.

Solution: Pat is correct. Indeed the theorem is as they state.
The problem with Sam's argument is that we must begin with an arbitrary open cover of X, not of Y. If $\mathcal{A}$ is an open cover of X, it is not true that

$$\mathcal{C} = \{C = f(A) \mid A \in \mathcal{A}\}$$

is an open cover of Y, since $f(A)$ might not be open even if A is open! (What's more, if f is not surjective then this only covers $f(X)$, not Y.) Therefore we have no way to make sure that an open cover of X "comes from" an open cover of Y, and so we have no way to ensure that general open cover of X has a finite subcover, and the compactness of Y is immaterial to the compactness of X.