Name:

Problem 1: *For each of the following, say if it is true or false. No justification is needed.*

(a) The space $\mathbb{R}$ in the usual topology is compact.

This is false. The open cover

$$\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}_+\}$$

has no finite subcover.

(b) Since $\mathbb{R}$ is Hausdorff and $Y = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ is compact, Y is closed in $\mathbb{R}$.

This is true. In a Hausdorff space every compact subspace is closed. (Earlier, we would have proved that Y is closed by proving that it contains all of its limit points, or by showing that it is the closure of the set $\{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$.)

(c) In a compact space, every compact subspace is closed.

This is false. Consider the set $X = \{x_1, x_2\}$ with the trivial topology $\mathcal{T} = \{\varnothing, X\}$. Then the subspace $\{x_1\}$ is compact, since its only open cover is X and this is a finite cover, and therefore a finite subcover of itself already. However, $\{x_1\}$ is not closed because $\{x_2\}$ is not open. Alternatively, consider X any infinite set in the finite complement topology. Then every subspace of X is compact, but only X and its finite subspaces are closed.

(d) Let X be a topological space, with subspace $Y \subset X$. To determine if Y is compact, we can consider whether every open cover of Y has a finite subcover, where here "open" can mean open in Y or open in X. However, to determine if Y is connected, we must decide if it admits a separation into two disjoint and nonempty sets that are open in Y, and this condition cannot be replaced by sets that are open in X.

This is true! The first statement about compactness is Lemma 26.1 and the definition of compactness. The second statement first gives (correctly) the definition of connectedness. We now show that we cannot say that Y is a connected subspace of X if and only if there is no pair of sets U, V disjoint and whose intersection with Y is nonempty, open in X, such that $Y \subset U \cup V$. Indeed, consider $X = \{x_1, x_2, x_3\}$ with the topology $\mathcal{T} = \{\varnothing, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, X\}$. Then $Y = \{x_1, x_3\}$ is not connected, since $\{x_1\}$ and $\{x_3\}$ are open in Y and form a separation, but there are not U, V open in X, nonempty and disjoint with $Y \subset U \cup V$.