Math 295 - Spring 2020
Homework 9

This homework is due on Wednesday, March 18. Most problems are adapted from Munkres's *Topology*.

1. Let X be a *finite* set, and let d be any metric on X. Show that the topology induced by d is the discrete topology.

2. Now let X be a set of arbitrary size. Show that the discrete metric (given in Example 1 on page 120) induces the discrete topology on X.
   Note: From Problem 1, we remark that this statement is only interesting when X is *infinite*.

3. (a) Let
   $$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|$$
   be a metric on $\mathbb{R}^n$, where $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. (Note that we have shown that this is a metric in Homework 7, problem 2.) Show that d' induces the usual topology on $\mathbb{R}^n$. Sketch a basis element when $n = 2$.

   (b) We generalize d' in the following way: Given $p \geq 1$, let
   $$d_p(\mathbf{x}, \mathbf{y}) = \left(|x_1 - y_1|^p + \cdots + |x_n - y_n|^p\right)^{1/p},$$
   where $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. For this problem, you may assume that d_p is a metric on $\mathbb{R}^n$. (Note that the graduate students have shown this on Homework 7.) Show that d_p induces the usual topology on $\mathbb{R}^n$.

4. In this problem we will show that the Euclidean metric d given by
   $$d(\mathbf{x}, \mathbf{y}) = d_2(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$
   on $\mathbb{R}^n$ is indeed a metric. For this proof we will need the following operations on $\mathbb{R}^n$: For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$:
   $$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n, \\ c\mathbf{x} &= (cx_1, cx_2, \ldots, cx_n) \in \mathbb{R}^n, \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \in \mathbb{R}, \\ \|\mathbf{x}\| &= \left(x_1^2 + \cdots + x_n^2\right)^{1/2}. \end{aligned}$$

   (a) Show that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.

   (b) Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$.
   Hint: If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.

(c) Show that $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
Hint: Compute $(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y})$ and apply (b).

(d) Verify that d is a metric.

Extra problem for graduate credit:

1. Let X be a metric space with a metric d and topology $\mathcal{T}$ induced by d.

   (a) Show that $d\colon X \times X \to \mathbb{R}$ is continuous, where $\mathbb{R}$ has the usual topology.

   (b) Let X' be a topological space having the same underlying set as X. (In other words, X' is the same set as X, but considered with a different topology $\mathcal{T}'$, which doesn't necessarily come from any metric.) Show that if $d\colon X' \times X' \to \mathbb{R}$ is continuous, then $\mathcal{T} \subset \mathcal{T}'$.