Math 295 - Spring 2020
Solutions to Homework 6
Review Homework for Exam 1

1. We check the axioms defining an order relation:

   - (Comparability) Let $a_1 \times b_1 \neq a_2 \times b_2 \in A \times B$. If $a_1 \neq a_2$, since A is a simply ordered set we must have $a_1 <_A a_2$ or $a_2 <_A a_1$ (by comparability for $<_A$), in which case $a_1 \times b_1 < a_2 \times b_2$, or $a_2 \times b_2 < a_1 \times b_1$, respectively. If $a_1 = a_2$, then $b_1 \neq b_2$, so $b_1 <_B b_2$ or $b_2 <_B b_1$ by comparability of $<_B$, and then once again, $a_1 \times b_1 < a_2 \times b_2$, or $a_2 \times b_2 < a_1 \times b_1$, respectively.
   - (Nonreflexivity) It is true that there is no $a \times b$ such that $a \times b < a \times b$. Indeed, this would imply either $a <_A a$, which violates nonreflexivity of $<_A$, or $b <_B b$, which violates nonreflexivity of $<_B$.
   - (Transitivity) If $a_1 \times b_1 < a_2 \times b_2$ and $a_2 \times b_2 < a_3 \times b_3$, then we can consider four cases:
     - If $a_1 <_A a_2$ and $a_2 <_A a_3$, then $a_1 <_A a_3$ by transitivity of $<_A$, so $a_1 \times b_1 < a_3 \times b_3$.
     - If $a_1 <_A a_2$ and $a_2 = a_3$, then $a_1 <_A a_3$, so $a_1 \times b_1 < a_3 \times b_3$.
     - If $a_1 = a_2$ and $a_2 <_A a_3$, then $a_1 <_A a_3$, so $a_1 \times b_1 < a_3 \times b_3$.
     - If $a_1 = a_2$ and $a_2 = a_3$, then $b_1 <_B b_2$ and $b_2 <_B b_3$, so $b_1 <_B b_3$ by transitivity of $<_B$, and $a_1 \times b_1 < a_3 \times b_3$.

     In any case, $a_1 \times b_1 < a_3 \times b_3$.

2. First, let $A \subset X$. Then $\pi_1^{-1}(A) = A \times Y$ (since anything with first coordinate in A is mapped to A under π_1), and $\pi_1(A \times Y) = A$. Therefore $\pi_1(\pi_1^{-1}(A)) = A$ as claimed.

   Now let $W \subset X \times Y$. Then

   $$\pi_1(W) = \{x \in X \mid \text{there is } y \in Y \text{ with } x \times y \in W\} \subset X.$$

   From our argument above, we thus have

   $$\pi_1^{-1}(\pi_1(W)) = \pi_1(W) \times Y.$$

   Therefore, in general we will have $W \subset \pi_1^{-1}(\pi_1(W))$, with equality only if whenever $x \times y \in W$, then $x \times \tilde{y} \in W$ for all $\tilde{y} \in Y$.

3. No, it is not. Suppose that J is an indexing set and $U_\alpha \in \mathcal{T} \cup \mathcal{T}'$ for all $\alpha \in J$. Now it is not true that $\cup_{\alpha \in J} U_\alpha \in \mathcal{T}$ necessarily, since for any one α it might be the case that $U_\alpha \in \mathcal{T}'$ but not in $\mathcal{T}$! Similarly, $\cup_{\alpha \in J} U_\alpha \notin \mathcal{T}'$ necessarily either. Therefore $\cup_{\alpha \in J} U_\alpha \notin \mathcal{T} \cup \mathcal{T}'$, and unions of elements of $\mathcal{T} \cup \mathcal{T}'$ might not belong to $\mathcal{T} \cup \mathcal{T}'$.

4. In the subspace topology, for each $n \in \mathbb{Z}_+$, the open interval $(n-1, n+1) \subset \mathbb{R}$ is such that $\mathbb{Z}_+ \cap (n-1, n+1) = \{n\}$. As a consequence, every single point set $\{n\}$ is open, so this is the discrete topology.

5. (a) The closed sets of X are exactly the finite subsets (including the empty set) and X.

   (b) Every set is closed in the discrete topology.

6. Let $x \in \overline{A}$. Then every neighborhood of x intersects A. But as $A \subset B$, this point of intersection is necessarily contained in B as well, so every neighborhood of x intersects B, and $x \in \overline{B}$.

7. Let U be a neighborhood of 0. By definition of a basis of open sets, there is a basis element for the order topology, given by an open interval (a, b), such that $0 \in (a, b) \subset U$. We show that $U \cap (0, 1]$ is nonempty by showing that $(a, b) \cap (0, 1]$ is nonempty. Indeed, since $0 < b$, there is x such that $0 < x < b$, so that $x \in (a, b)$ and $x \in (0, 1]$. Therefore $x \in (a, b) \cap (0, 1] \subset U \cap (0, 1]$ and $U \cap (0, 1]$ is nonempty for every neighborhood U of 0.

8. Let $x \neq y \in X$. Since Δ is closed, Δ is its own closure, and $x \times y \notin \overline{\Delta}$. Therefore there is an open set W of $X \times X$ such that $x \times y \in W$ and $W \cap \Delta = \varnothing$. By the definition of a basis for the product topology on $X \times X$, there is a basis element of the form $U \times V$, with U and V open in X, such that $x \times y \in U \times V \subset W$. Now, since $W \cap \Delta = \varnothing$, certainly $(U \times V) \cap \Delta = \varnothing$. It follows from this that $U \cap V = \varnothing$. (If $z \in U \cap V$, then $z \times z \in (U \times V) \cap \Delta$.) But since $x \times y \in U \times V$, we have that $x \in U$ and $y \in V$ with $U \cap V = \varnothing$ and U, V open, so X is Hausdorff.

9. Let $U \subset X$ be open. Then $i^{-1}(U) = U \cap Y$. This is open in Y by definition of the subspace topology, so i is continuous.