Math 295 - Spring 2020
Solutions to Homework 5

1. (a) Since A is closed in Y, there is V open in Y such that $A = Y - V$. But since Y has the subspace topology, there is U open in X such that $V = U \cap Y$. Since Y is closed in X, there is W open in X such that $Y = X - W$. Putting all this together, we have
$$A = (X - W) - (U \cap Y) = X - (W \cup U).$$
Since $W \cup U$ is a union of two open sets in X, it is open in X, so A is closed in X.

   (b) Since A is closed in X, there is U open in X such that $A = X - U$, and since B is closed in Y, there is V open in Y such that $B = Y - V$. Then
$$A \times B = X \times Y - (U \times Y \cup X \times V).$$
Since $U \times Y$ and $X \times V$ are open in $X \times Y$ by definition of the product topology, and the union of two open sets is open, we have that $A \times B$ is closed in $X \times Y$.

2. Let $x \in \cup_{\alpha \in J} \bar{A}_\alpha$. Then there is $\alpha \in J$ such that $x \in \bar{A}_\alpha$, which means that every neighborhood of x intersects A_α. In turn, since $A_\alpha \subset \cup_{\alpha \in J} A_\alpha$, this means that every neighborhood of x intersects $\cup_{\alpha \in J} A_\alpha$, so $x \in \overline{\cup_{\alpha \in J} A_\alpha}$.

   An example where the inclusion is strict is given by $J = \mathbb{Z}_+$ and $A_n = \{\frac{1}{n}\} \subset \mathbb{R}$, where $\mathbb{R}$ is given the standard topology. Then $\bar{A}_n = \{\frac{1}{n}\}$ since $\mathbb{R}$ is Hausdorff and one-point sets are closed. It follows that
$$\bigcup_{n=1}^{\infty} \bar{A}_n = \bigcup_{n=1}^{\infty} A_n.$$
However, $\overline{\bigcup_{n=1}^{\infty} A_n} \neq \bigcup_{n=1}^{\infty} A_n$, since $0 \in \overline{\bigcup_{n=1}^{\infty} A_n} - \bigcup_{n=1}^{\infty} A_n$: Indeed, any open set $U \subset \mathbb{R}$ containing 0 must contain a basis element (an interval) about 0, so $0 \in (a, b) \subset U$ for some $a, b \in \mathbb{R}$. Then we know that there is $n \in \mathbb{Z}_+$ such that $\frac{1}{n} \in (a, b)$, so U intersects $\bigcup_{n=1}^{\infty} A_n$.

3. (a) Let X and Y be Hausdorff spaces, and let $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$. Without loss of generality, $x_1 \neq x_2$. Then since X is Hausdorff, there are $U_1 \ni x_1$ and $U_2 \ni x_2$, where U_1, U_2 are open in X, such that $U_1 \cap U_2 = \varnothing$. Then the sets $W_1 = U_1 \times Y$ and $W_2 = U_2 \times Y$ are open in $X \times Y$, they are disjoint (since no point can have first coordinate in U_1 and U_2 at once) and $(x_1, y_1) \in W_1$ and $(x_2, y_2) \in W_2$. So $X \times Y$ is Hausdorff.

   (b) Let X be a Hausdorff space and Y be a subspace of X. Let $y_1 \neq y_2 \in Y$. Since $Y \subset X$, it follows that $y_1 \neq y_2 \in X$ as well, and since X is Hausdorff, there are $U_1 \ni x_1$ and $U_2 \ni x_2$, where U_1, U_2 are open in X, such that $U_1 \cap U_2 = \varnothing$. But then $V_1 = U_1 \cap Y$ and $V_2 = U_2 \cap Y$ are open in Y and disjoint, and $y_1 \in V_1$, $y_2 \in V_2$, so Y is Hausdorff.

4. (a) Suppose that i is continuous. Then for all $V \in \mathcal{T}$, we have that $i^{-1}(V) = V$ is open in X' since i is continuous, so $V \in \mathcal{T}'$, and $\mathcal{T} \subset \mathcal{T}'$.

   Suppose now that $\mathcal{T} \subset \mathcal{T}'$. Then for $V \in \mathcal{T}$, $i^{-1}(V) = V \in \mathcal{T}'$, so the inverse image of an open set is open and i is continuous.

   (b) We assume the result of part (a), so all that remains to show is that i^{-1} is continuous if and only if $\mathcal{T}' \subset \mathcal{T}$. But actually $i^{-1} = i$, so this is part (a) but with $i\colon X \to X'$ and the result follows.

5. Before we begin, we will need a lemma: Let $g\colon \mathbb{R} \to \mathbb{R}$ be a linear function; in other words it is given by $g(x) = mx + b$ for $m, b \in \mathbb{R}$. Then g is a homeomorphism. (Here $\mathbb{R}$ has the standard topology.)

   First we show that g is bijective. Indeed, $g^{-1}(x) = \frac{x-b}{m}$; one shows quickly that g^{-1} is a well-defined function and $g \circ g^{-1} = g^{-1} \circ g = 1$.

   To show that g is continuous, it is enough to show that if $r_1 < r_2 \in \mathbb{R}$, $g^{-1}((r_1, r_2))$ is open, since the open intervals form a basis for the open sets of $\mathbb{R}$. Since $g^{-1}(x) = \frac{x-b}{m}$, $g^{-1}((r_1, r_2))$ is the values that the expression $\frac{x-b}{m}$ takes if $r_1 < x < r_2$:

$$r_1 < x < r_2 \iff r_1 - b < x - b < r_2 - b \iff \frac{r_1 - b}{m} < \frac{x-b}{m} < \frac{r_2 - b}{m}.$$

   So

$$g^{-1}((r_1, r_2)) = \left(\frac{r_1 - b}{m}, \frac{r_2 - b}{m} \right),$$

   which is open.

   Now we can easily show the result we seek: First we prove that since $(0,1)$, (a,b), $[0,1]$ and $[a,b]$ are all subspaces of $\mathbb{R}$, it is enough to show that $f(x) = \frac{x-a}{b-a}$ is a bijection from (a,b) to $(0,1)$ and from $[a,b]$ to $[0,1]$. Indeed, granting this, it only remains to show that f and f^{-1} are continuous. Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \frac{x-a}{b-a}$. By our lemma above, g is bijective and both g and g^{-1} are continuous. Applying Theorem 18.2(d) to restrict the domain of g to (a,b) and to $[a,b]$, we obtain that f is continuous on (a,b) and on $[a,b]$, and restricting the domain of g^{-1} to $(0,1)$ and to $[0,1]$ we obtain that f^{-1} is continuous on $(0,1)$ and on $[0,1]$. So the continuity conditions will follow once we have bijectivity.

   Note that here it is not enough to show that f^{-1} exists; the question here is whether the image of the set $[a,b]$ by f is $[0,1]$, and the image of (a,b) is $(0,1)$. In other words, upon restricting the domain, a function only remains bijective if its image is suitably restricted, so it is exactly the image of the new domain.

   But we have that

$$a \le x \le b \iff 0 \le x - a \le b - a \iff 0 \le \frac{x-a}{b-a} \le 1,$$

   and the same holds with the inequalities replaced by strict inequalities, so the result follows.

Extra problems for graduate credit:

1. Let X be a topological space and suppose that $\{x\}$ is closed for all $x \in X$. (This is equivalent to the T_1-axiom, since finite unions of closed sets are closed.) Let $x_1 \neq x_2 \in X$. Since $\{x_1\}$ is closed, $x_2 \notin \overline{\{x_1\}}$, so there is a neighborhood U of x_2 such that $U \cap \{x_1\} = \varnothing$. Therefore $x_1 \notin U$, and x_2 has a neighborhood that doesn't contain x_1. The same argument can be applied with the roles of x_1 and x_2 reversed, so x_1 also has a neighborhood that doesn't contain x_2.

   Now suppose that for each pair of points of X, each has a neighborhood not containing the other. Let $x \in X$, and suppose that $y \in \overline{\{x\}}$. If $y \neq x$, then y has a neighborhood U that does not contain x, and for that neighborhood it is the case that $U \cap \{x\} = \varnothing$, yielding a contradiction (y cannot be in the closure of $\{x\}$). Therefore $y \in \overline{\{x\}}$ implies that $y = x$ and $\{x\}$ is closed, and therefore also every finite point set.

2. We can talk about this in person :)