Math 295 - Spring 2020
Homework 5

This homework is due on Friday, February 14. All problems are adapted from Munkres's *Topology*.

1. (a) Show that if A is closed in Y and Y is closed in X, then A is closed in X.
   (b) Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.
2. Let J be an indexing set, and A_α be subsets of a space X for $\alpha \in J$. Prove that

$$\bigcup_{\alpha \in J} \bar{A}_\alpha \subset \overline{\bigcup_{\alpha \in J} A_\alpha},$$

   and give an example where equality fails.
3. (a) Show that the product of two Hausdorff spaces is Hausdorff.
   (b) Show that a subspace of a Hausdorff space is Hausdorff.
4. Let X and X' denote the same set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i \colon X' \to X$ be the identity function.
   (a) Show that i is continuous if and only if $\mathcal{T} \subset \mathcal{T}'$.
   (b) Show that i is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.
5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ is homeomorphic with $[0, 1]$.

Extra problems for graduate credit:

1. Please read the definition of the T_1 *axiom* on page 99 of the book, and Theorem 17.9 immediately following it. Then, show that the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.
2. (This problem is difficult. Please explore and try to get the answer as best you can, and write up what you find out. This problem is due to Kuratowski.) Consider the collection of all subsets A of the topological space X, the *power set of* X; $A \in \mathcal{P}(X)$ if and only if $A \subset X$. The operations of closure $A \to \bar{A}$ and complementation $A \to X - A$ are functions from $\mathcal{P}(X)$ to itself.
   (a) Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.
   (b) Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.