Math 295 - Spring 2020
Solutions to Homework 14

1. Suppose for a contradiction that they are homeomorphic, and let $f\colon (0,1] \to (0,1)$ be a homeomorphism. Let $a \in (0,1)$ be such that $f(1) = a$. Then because f is a bijection, $f^{-1}(a) = 1$ (no other point maps to a).

   By restriction of the domain (Theorem 18.2(d)), the function $f|_{(0,1)}\colon (0,1) \to (0,1)$ is also continuous. We also have that the image set of $f|_{(0,1)}$ is contained in the set $(0,a)\cup(a,1)$, since now that 1 has been removed from the domain, nothing maps to the value a. By restriction of the range (Theorem 18.2(e)), $f|_{(0,1)}\colon (0,1) \to (0,a)\cup(a,1)$ is also continuous. In addition, it is surjective (in fact, it is bijective). But we know that the image of a connected set under a continuous map is connected, and therefore we get a contradiction since $(0,a)\cup(a,1)$ is not connected.

   Therefore $(0,1]$ and $(0,1)$ are not homeomorphic.

2. Let $Y \subset X$ be a nonempty subspace, and suppose that it is covered by an arbitrary collection $\mathcal{A}$ of sets open in X. Let $A \in \mathcal{A}$ be any nonempty element of this collection. Since A is open and nonempty, its complement $X - A$ is finite. Since $Y - A \subset X - A$, $Y - A$ is also finite, say $Y - A = \{y_1, y_2, \dots, y_n\}$. For each i, let $A_i \in \mathcal{A}$ be such that $y_i \in A_i$. Then the finite subcollection $\{A, A_1, \dots, A_n\}$ covers Y, and Y is compact.

3. (a) If X is compact in the $\mathcal{T}'$ topology, then it is compact in the $\mathcal{T}$ topology. Indeed, let $\mathcal{A}$ be a collection of sets that belong to $\mathcal{T}$ and that cover X. Then they also belong to $\mathcal{T}'$, and since X is compact in this topology, there is a finite subcover. Therefore X is compact in the $\mathcal{T}$ topology.

   However, if X is compact in the $\mathcal{T}$ topology, X may or may not be compact in the $\mathcal{T}'$ topology. We give two examples: First let $X = \mathbb{R}$, $\mathcal{T}'$ be the usual topology and $\mathcal{T}$ be the trivial topology. Since the trivial topology is contained in every topology, $\mathcal{T}' \supset \mathcal{T}$. Then X is compact in the $\mathcal{T}$ topology (every space is compact in the trivial topology) but it is not compact in the $\mathcal{T}'$ topology (we showed this in class).

   However, if $X = \mathbb{R}$ still but now $\mathcal{T}'$ is the finite complement topology, and again $\mathcal{T}$ is the trivial topology. Since the trivial topology is contained in every topology, once again $\mathcal{T}' \supset \mathcal{T}$. This time X is compact in both topologies, by problem 2 of this homework.

   (b) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X such that X is compact and Hausdorff in both topologies. Furthermore, suppose that $\mathcal{T}' \supsetneq \mathcal{T}$. We will derive a contradiction. Note that this will complete the proof: by symmetry, this will show that if we suppose that $\mathcal{T}' \subsetneq \mathcal{T}$, we also get a contradiction. If both strict inclusions lead to contradictions, then it follows that it must be the case that $\mathcal{T}' = \mathcal{T}$ or $\mathcal{T}$ and $\mathcal{T}'$ are not comparable.

   We thus proceed with our work. Let $U \in \mathcal{T}' - \mathcal{T}$. Then the set $A = X - U$ is closed in $\mathcal{T}'$ but not in $\mathcal{T}$. Since X is compact in the $\mathcal{T}'$ topology, A is compact in the $\mathcal{T}'$ topology. By part (a) of this problem, A is then also compact in the $\mathcal{T}$ topology. Since X is Hausdorff in the $\mathcal{T}$ topology, A is therefore closed in X. This is the contradiction.

4. Let $Y_1, \ldots, Y_n$ be compact, and suppose that they are all contained in a topological space X. Let $\mathcal{A}$ be a collection of sets open in X, such that $\mathcal{A}$ covers $\bigcup_{i=1}^n Y_i$. Then for each i, $\mathcal{A}$ covers Y_i, and therefore there is a finite subcollection of elements of $\mathcal{A}$ that cover Y_i:
$$Y_i \subset \bigcup_{j=1}^{m_i} A_{i,j}.$$
Now consider the set
$$\{A_{1,1}, \ldots, A_{1,m_1}, A_{2,1}, \ldots, A_{2,m_2}, \ldots, A_{n,m_n}\}.$$
This is a finite collection of elements, since it is the union of finitely many finite collections. Furthermore, it covers $\bigcup_{i=1}^n Y_i$, and therefore $\bigcup_{i=1}^n Y_i$ is compact.

Extra problem for graduate credit:

1. As suggested, we first show that if A is closed in X with empty interior, and U is any nonempty open of X, then there is a nonempty open set V such that $\overline{V} \subset U$ and $A \cap \overline{V} = \varnothing$.

   To do this, we first remark that it cannot be the case that $U \subset A$, since A has empty interior and U is not empty. Therefore, there is $x \in X$ such that $x \in U - A$. We now note that since X is compact and $A \cup (X - U)$ is closed (this is a union of two closed sets), $A \cup (X - U)$ is compact. Since X is Hausdorff and $x \notin A \cup (X - U)$, there are open sets V, W in X such that $V \cap W = \varnothing$, $x \in V$ and $A \cup (X - U) \subset W$.

   We claim that V as described above is the open set we sought. Indeed V is nonempty and open. Suppose that $a \in A$, we show that $a \notin \overline{V}$, which will prove the last claim. Indeed, if $a \in A$, then $a \in W$ which is an open set, and $W \cap V = \varnothing$. Therefore there is a neighborhood of a that does not intersect V, and $a \notin \overline{V}$.

   Finally, we show that $\overline{V} \subset U$. Indeed, let $x \in X - U$, in the same manner as just above, we can show that $x \notin \overline{V}$, since $x \in W$ which does not intersect V.

   We now proceed to Step 2. Suppose by contradiction that $\bigcup_{n=1}^\infty A_n$ does not have empty interior. Then there is U nonempty and open such that $U \subset \bigcup_{n=1}^\infty A_n$, and consider A_1 which is closed with empty interior by hypothesis. Then by Step 1 there is a nonempty open set V_1 with $\overline{V}_1 \subset U \subset \bigcup_{n=1}^\infty A_n$ such that $A_1 \cap \overline{V}_1 = \varnothing$.

   Now let $U = V_1$, which is nonempty and open, and consider $A = A_1 \cup A_2$. A is the union of two closed sets, and therefore it is closed. We must show that its interior is empty. Let W be open in X such that $W \subset A$. Then $W \cap (X - A_2)$ is an open set, since $X - A_2$ is open, and $W \cap (X - A_2) \subset A_1$. Since A_1 has empty interior by hypothesis, $W \cap (X - A_2)$ is empty, but this implies $W \subset A_2$, which is a contradiction since A_2 also has empty interior. We note for the future that we have proved that a finite union of closed sets with empty interior has empty interior.

Going back to our set up though, we have $U = V_1$, which is nonempty and open, and $A = A_1 \cup A_2$, which is closed with empty interior. Therefore there is V_2 nonempty and open with $\overline{V}_2 \subset V_1$ and $A \cap \overline{V}_2 = \varnothing$.

We continue like this: At every step we can find V_n nonempty and open such that

$$(\bigcup_{i=1}^{n} A_n) \cap \overline{V}_n = \varnothing,$$

and

$$\bigcup_{n=1}^{\infty} A_n \supset \overline{V}_1 \supset \overline{V}_2 \supset \overline{V}_3 \supset \cdots \supset \overline{V}_{n-1} \supset V_n,$$

because $\bigcup_{i=1}^{n} A_n$ is closed with empty interior.

Continuing in this way, we have a nested sequence

$$\overline{V}_1 \supset \overline{V}_2 \supset \overline{V}_3 \supset \cdots \supset \overline{V}_n \supset \cdots$$

of nonempty closed sets in X. By Th eorem 26.9 there is a point in their intersection since X is compact. In other words, there is $x \in X$ such that $x \in \overline{V}_n$ for each n. In particular, this means that $x \notin A_n$ for any n, so $x \notin \bigcup_{n=1}^{\infty} A_n$. At the same time, since each $\overline{V}_n \subset \bigcup_{n=1}^{\infty} A_n$, certainly $x \in \bigcap_{n=1}^{\infty} \overline{V}_n \subset \bigcup_{n=1}^{\infty} A_n$, which is a contradiction.