Math 295 - Spring 2020
Solutions to Homework 13

1. (a) Yes, $X \times Y$ is path connected. The idea is the following, and a formal proof follows: Let $x \times y, x' \times y' \in X \times Y$. To make a path between these, we first make a path between $x \times y$ and $x' \times y$ in $X \times y$. This path exists because $X \times y$ is homeomorphic to X, and therefore path connected as well. Once we are at $x' \times y$, we can follow a path from $x' \times y$ to $x' \times y'$ in $x' \times Y$, which is also path connected. This gives a path from $x \times y$ to $x' \times y'$.

We now formalize this proof: Let $x, x' \in X$ and $y \in Y$. We show the existence of a path γ from $x \times y$ to $x' \times y$ in $X \times Y$. Let $\tilde{\gamma}\colon [a, b] \to X$ be a path from x to x'. This exists since X is path connected. Let $\phi\colon X \to X \times y$ be the map sending $x \mapsto x \times y$. We have shown in class that this is a homeomorphism, and so in particular it is continuous. Finally, let $\psi\colon X \times y \to X \times Y$ be the inclusion map, which is continuous by Theorem 18.2(b). We have thus that the composition $\psi \circ \phi \circ \tilde{\gamma}\colon [a, b] \to X \times Y$ is continuous, and a path from $\psi(\phi(\tilde{\gamma}(a))) = \psi(\phi(x)) = x \times y$ to $\psi(\phi(\tilde{\gamma}(b))) = \psi(\phi(x')) = x' \times y$. Therefore $\gamma = \psi \circ \phi \circ \tilde{\gamma}$ is a path from $x \times y$ to $x' \times y$.

Since the situation is completely symmetric, we can use the argument above to show that for $x' \in X$ and $y, y' \in Y$, there is also a path $\gamma'\colon [a', b'] \to X \times Y$ from $x' \times y$ to $x' \times y'$.

Given the existence of these two paths, we now prove the existence of a path from $x \times y$ to $x' \times y'$. Recall that we have a path $\gamma\colon [a, b] \to X \times Y$ which goes from $x \times y$ to $x' \times y$, and a path $\gamma'\colon [a', b'] \to X \times Y$ from $x' \times y$ to $x' \times y'$. What we want to do is "follow one path then the other," or concatenate these paths.

Let c be a real number strictly greater than b. Then we have shown already that the interval $[a', b']$ is homeomorphic to the interval $[b, c]$. (Technically, we showed that every closed interval is homeomorphic to $[0, 1]$, but since being homeomorphic is an equivalence relation, this means that every closed interval is homeomorphic to every other closed interval in $\mathbb{R}$.) In addition, in our proof we gave an order-preserving homeomorphism, which implies that we can further assume that this homeomorphism sends a' to b and b' to c. If $\phi\colon [b, c] \to [a', b']$ is one such order-preserving homeomorphism, by composing $\gamma' \circ \phi$, we obtain a path $\gamma''\colon [b, c] \to X \times Y$, still from $x' \times y$ to $x' \times y'$.

Now let $\eta\colon [a, c] \to X \times Y$ be given by the rule

$$\eta(t) = \begin{cases} \gamma(t) & \text{if } a \le t \le b, \text{ and} \\ \gamma''(t) & \text{if } b \le t \le c. \end{cases}$$

By the Pasting Lemma (Theorem 18.3), η is a continuous function. Indeed, if $A = [a, b]$ and $B = [b, c]$, then $X = [a, c]$ is indeed the union of the two closed subspaces A and B, γ and γ'' are continuous, and they agree on $A \cap B = \{b\}$:

$\gamma(b) = x' \times y = \gamma''(b)$. Therefore, η is a path in $X \times Y$ from $x \times y$ to $x' \times y'$, and $X \times Y$ is path connected.

(b) No, $\overline{A}$ is not necessarily path connected. Let

$$A = \{x \times \sin\left(\frac{1}{x}\right) \mid x > 0\} \subset \mathbb{R}^2.$$

Then A is path connected, but as shown in the book on page 157, in Example 7 of Section 24, $\overline{A}$ is not path connected. (A is denoted S in the book, it is the topologist's sine curve which we discussed in class.)

To show that A is path connected, let $x \times \sin\left(\frac{1}{x}\right)$ and $x' \times \sin\left(\frac{1}{x'}\right)$ be two elements of A. Then the map $\gamma\colon [x, x'] \to A$ given by $\gamma(t) = t \times \sin\left(\frac{1}{t}\right)$ is a path from $x \times \sin\left(\frac{1}{x}\right)$ to $x' \times \sin\left(\frac{1}{x'}\right)$, since it is the Cartesian product of two continuous maps (this is Theorem 18.4).

The proof that $\overline{A}$ is not path connected is on page 157 of the book, in Example 7, and we do not repeat it here.

(c) Yes, $f(X)$ is path connected. Let $c, d \in f(X)$, then there is $x \in X$ such that $f(x) = c$ and $y \in X$ such that $f(y) = d$. Let $\gamma\colon [a, b] \to X$ be a path from x to y. I claim that $f \circ \gamma\colon [a, b] \to Y$ is a path from c to d.

Indeed, we have that $(f \circ \gamma)(a) = f(\gamma(a)) = f(x) = c$, and $(f \circ \gamma)(b) = f(\gamma(b)) = f(y) = d$. Furthermore, both f and γ are continuous, and therefore their composition $f \circ \gamma$ is continuous.

(d) Yes, $\bigcup A_\alpha$ is path connected. The idea of the proof is the following, and a formal proof follows: Let $x, y \in \bigcup A_\alpha$. Note that then there is δ such that $x \in A_\delta$ and ε such that $y \in A_\varepsilon$. Then if $z \in \bigcap A_\alpha$, in particular $z \in A_\delta$ and $z \in A_\varepsilon$. We can then get a path from x to y by first taking a path from x to z in A_δ (which exists since A_δ is path connected) and then a path from z to y in A_ε, which overall gives a path from x to y in $\bigcup A_\alpha$.

We now formalize this proof. With notation as above, let $\gamma\colon [a, b] \to A_\delta$ be a path from x to z in A_δ. Then by expanding the range (Theorem 18.2(e)), we have that $\gamma\colon [a, b] \to \bigcup A_\alpha$ is also continuous (we abuse notation here and keep the name γ for this new function) and therefore a path from x to z in $\bigcup A_\alpha$.

Similarly, let $\gamma'\colon [b, c] \to A_\varepsilon$ be a path from z to y in A_ε. We note that we may assume that the domain is the interval $[b, c]$ since all closed intervals are homeomorphic via an order-preserving homeomorphism. Again, by expanding the range, we obtain a path $\gamma'\colon [b, c] \to \bigcup A_\alpha$ from z to y.

Now we once again concatenate the paths to obtain $\eta\colon [a, c] \to \bigcup A_\alpha$ given by

$$\eta(t) = \begin{cases} \gamma(t) & \text{if } a \leq t \leq b, \text{ and} \\ \gamma'(t) & \text{if } b \leq tc. \end{cases}$$

This is continuous by the Pasting Lemma, and therefore a path from x to y in $\bigcup A_\alpha$.