Math 259 - Spring 2019
Homework 2 Solutions

1. (a) 10 (b) 110

2. (a) 8 (b) 5

3. Omitted

4. (a) $|5| = \sqrt{5^2} = 5$
   (b) $|-4| = \sqrt{(-4)^2} = 4$
   (c) $|2i| = \sqrt{2^2} = 2$
   (d) $|3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$
   (e) $|-1+3i| = \sqrt{(-1)^2+3^2} = \sqrt{10}$
   (f) $\left|\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1$

5. (a) This would be a 1-qbit superposition. What must be true for that to be the case is that $\left|\frac{3i}{5}\right|^2 + \left|\frac{4}{5}\right|^2 = 1$. Indeed we have

$$\left|\frac{3i}{5}\right|^2 + \left|\frac{4}{5}\right|^2 = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1.$$

   So this is a valid 1-qbit superposition. The probability of observing $|0\rangle$ is $\left|\frac{3i}{5}\right|^2 = \frac{9}{25}$ and the probability of observing $|1\rangle$ is $\left|\frac{4}{5}\right|^2 = \frac{16}{25}$.

   (b) This also would be a 1-qbit superposition. We need that $\left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = 1$. However, that is not the case:

$$\left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \neq 1.$$

   Therefore this is not a valid superposition.

   (c) This would be a 3-qbit superposition. For this to be true, we need $\sum_{k=0}^{7} \left|\frac{1}{2\sqrt{2}}\right|^2 = 1$. Indeed we have

$$\sum_{k=0}^{7} \left|\frac{1}{2\sqrt{2}}\right|^2 = 8\left(\frac{1}{2\sqrt{2}}\right)^2 = 8\left(\frac{1}{4\cdot 2}\right) = 8\cdot\frac{1}{8} = 1.$$

   So this is a valid 3-qbit superposition. The probability of observing the states $|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle$ and $|111\rangle$ are each $\left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{8}$.

(d) This would be a general n-qbit superposition. For this to be true this time we would need $\sum_{k=0}^{2^n-1} \left|\frac{1}{\sqrt{2^n}}\right|^2 = 1$. Indeed we have

$$\sum_{k=0}^{2^n-1} \left|\frac{1}{\sqrt{2^n}}\right|^2 = 2^n \left(\frac{1}{\sqrt{2^n}}\right)^2 = 2^n \left(\frac{1}{2^n}\right) = 1.$$

So this is again a valid superposition. The probability of observing any of the 2^n possible states $|00\ldots0\rangle$ through $|11\ldots1\rangle$, as they range through all possibilities for n-bit binary numbers, are each $\left|\frac{1}{\sqrt{2^n}}\right|^2 = \frac{1}{2^n}$.

6. (a) We will use the following notation: To show that we have applied the Hadamard gate to the first qbit, and that this controls the outcome of the first qbit of the answer, we will write

$$|0\cdot\rangle \mapsto \frac{1}{\sqrt{2}}|0\cdot\rangle + \frac{1}{\sqrt{2}}|1\cdot\rangle.$$

When we separately apply the Hadamard gate to the second qbit, and this controls the outcome of the second qbit of the answer, we will get

$$|\cdot 0\rangle \mapsto \frac{1}{\sqrt{2}}|\cdot 0\rangle + \frac{1}{\sqrt{2}}|\cdot 1\rangle.$$

Now to get $|00\rangle$ as the outcome, we must get $|0\cdot\rangle$ and also $|\cdot 0\rangle$. To get $|01\rangle$ as the outcome, we must get $|0\cdot\rangle$ and $|\cdot 1\rangle$, etc. Since we are applying the Hadamard gate to the first and the second qbit independently, the coefficients of each outcome (in the first and the second qbit) get multiplied to get the coefficient of the overall outcome. This gives us:

$$\begin{aligned}
|00\rangle &\mapsto (\frac{1}{\sqrt{2}}|0\cdot\rangle + \frac{1}{\sqrt{2}}|1\cdot\rangle) \otimes (\frac{1}{\sqrt{2}}|\cdot 0\rangle + \frac{1}{\sqrt{2}}|\cdot 1\rangle) \\
&\mapsto \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}|11\rangle \\
&\mapsto \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle
\end{aligned}$$

(We use here $\otimes$ for our multiplication since this is the correct "kind" of multiplication to use here, even though we haven't talked about it. Might as well get the notation right.)

So indeed, if there is a way to apply the Hadamard gate to each qbit of a 2-qbit superposition separately (there is!) then when the superposition $|00\rangle$ is entered, the superposition $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$ will come out. Yay!

After we have applied this new gate, the probability of observing $|00\rangle, |01\rangle, |10\rangle$ or $|11\rangle$ will each be $\left|\frac{1}{2}\right|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. So we have gone from a superposition with no uncertainty (it was $|00\rangle$ with probability 1) to a superposition where each outcome is equally likely. This is handy.

(b) But what does the gate do to other superpositions? What if we enter $|01\rangle, |10\rangle$ or $|11\rangle$? Or a superposition of these four states? The gate can be applied to so much more than just the input $|00\rangle$!

We compute what the gate does by first computing what it does to $|01\rangle, |10\rangle$ and $|11\rangle$, and then extending linearly to a superposition $a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$. Throughout we will use that

$$\begin{aligned}
|0\cdot\rangle &\mapsto \frac{1}{\sqrt{2}}|0\cdot\rangle + \frac{1}{\sqrt{2}}|1\cdot\rangle, \\
|\cdot 0\rangle &\mapsto \frac{1}{\sqrt{2}}|\cdot 0\rangle + \frac{1}{\sqrt{2}}|\cdot 1\rangle, \\
|1\cdot\rangle &\mapsto \frac{1}{\sqrt{2}}|0\cdot\rangle - \frac{1}{\sqrt{2}}|1\cdot\rangle, \\
|\cdot 1\rangle &\mapsto \frac{1}{\sqrt{2}}|\cdot 0\rangle - \frac{1}{\sqrt{2}}|\cdot 1\rangle.
\end{aligned}$$

Then we have:

$$\begin{aligned}
|01\rangle &\mapsto (\frac{1}{\sqrt{2}}|0\cdot\rangle + \frac{1}{\sqrt{2}}|1\cdot\rangle) \otimes (\frac{1}{\sqrt{2}}|\cdot 0\rangle - \frac{1}{\sqrt{2}}|\cdot 1\rangle) \\
&\mapsto \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle,
\end{aligned}$$

and also

$$\begin{aligned}
|10\rangle &\mapsto (\frac{1}{\sqrt{2}}|0\cdot\rangle - \frac{1}{\sqrt{2}}|1\cdot\rangle) \otimes (\frac{1}{\sqrt{2}}|\cdot 0\rangle + \frac{1}{\sqrt{2}}|\cdot 1\rangle) \\
&\mapsto \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle,
\end{aligned}$$

and finally

$$\begin{aligned}
|11\rangle &\mapsto (\frac{1}{\sqrt{2}}|0\cdot\rangle - \frac{1}{\sqrt{2}}|1\cdot\rangle) \otimes (\frac{1}{\sqrt{2}}|\cdot 0\rangle - \frac{1}{\sqrt{2}}|\cdot 1\rangle) \\
&\mapsto \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle.
\end{aligned}$$

By linearity, we then have that

$$\begin{aligned}
a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle \mapsto\ & a_0(\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle) \\
& + a_1(\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle) \\
& + a_2(\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle) \\
& + a_3(\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle) \\
\mapsto\ & \frac{1}{2}(a_0 + a_1 + a_2 + a_3)|00\rangle \\
& + \frac{1}{2}(a_0 - a_1 + a_2 - a_3)|01\rangle \\
& + \frac{1}{2}(a_0 + a_1 - a_2 - a_3)|10\rangle \\
& + \frac{1}{2}(a_0 - a_1 - a_2 + a_3)|11\rangle.
\end{aligned}$$

This last expression describes the gate completely; it says what happens to any superposition that enters the gate.

Now we simply must write the matrix that sends

$$\begin{aligned}
(a_0, a_1, a_2, a_3) \mapsto \Big(&\frac{1}{2}(a_0 + a_1 + a_2 + a_3), \frac{1}{2}(a_0 - a_1 + a_2 - a_3), \\
&\frac{1}{2}(a_0 + a_1 - a_2 - a_3), \frac{1}{2}(a_0 - a_1 - a_2 + a_3)\Big).
\end{aligned}$$

This is the matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

This is our new gate!

(c) We would do the exact same thing, but with three qbits.

Sparing you the details, we would get the matrix

$$\frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

7. (a)

input	output
00	00
01	10
10	01
11	11

(b) There are two ways to solve this part. Either you just kind of have a good feeling for those things and "just know" that the matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

or it might be easier to first answer part (c) and say, yes, this is a quantum gate (see part (c) for why), and say that this would send the superposition

$$\begin{aligned} a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle &\mapsto a_0|00\rangle + a_1|10\rangle + a_2|01\rangle + a_3|11\rangle \\ &\mapsto a_0|00\rangle + a_2|01\rangle + a_1|10\rangle + a_3|11\rangle. \end{aligned}$$

So we want the matrix that sends

$$(a_0, a_1, a_2, a_3) \mapsto (a_0, a_2, a_1, a_3),$$

which is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

I'm not sure what's easiest, it depends on what one is comfortable with.

(c) In any case, yes, this gate is a quantum gate!

- It is reversible. We can see this either by looking at the list of input/outputs (each output appears exactly once, so we can reverse the gate, or in a more fancy way we can see that the gate gives a bijection), by reasoning that the gate must be reversible because it is its own inverse (we can swap the bits back!) or by writing the matrix and seeing that the matrix is invertible (which we can see by computing its determinant for example, or seeing that no row/column is a linear combination of the others). Anyway, it is reversible.
- It also sends superpositions with $|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 1$ to superpositions with the same property. This is not difficult to see, since

$$|a_0|^2 + |a_2|^2 + |a_1|^2 + |a_3|^2 = |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 1.$$

8. (a) This is of the form $re^{i\theta}$ with $r = 4$ and $\theta = \frac{2\pi}{3}$. Then

$$x = 4\cos\left(\frac{2\pi}{3}\right) = 4 \cdot -\frac{1}{2} = -2,$$

and

$$y = 4\sin\left(\frac{2\pi}{3}\right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3},$$

so $z = 2 + 2\sqrt{3}i$.

(b) This is of the form $z = x + iy$ with $x = -1$ and $y = -1$, so

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}.$$

Now to find θ is it tempting to just type in $\theta = \arctan(1)$, but that will output $\frac{\pi}{4}$. If we graph the point though, we see that in fact $\theta = \frac{5\pi}{4}$. So $z = \sqrt{2}e^{\frac{5\pi i}{4}}$.

(c) For this complex number we have $r = 2$ and $\theta = \frac{7\pi}{4}$, so

$$x = 2\cos\left(\frac{7\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2},$$

and

$$y = 2\sin\left(\frac{7\pi}{4}\right) = 2 \cdot -\frac{\sqrt{2}}{2} = -\sqrt{2},$$

so $z = \sqrt{2} - \sqrt{2}i$.

(d) This complex number has $x = 0$ and $y = 5$, so

$$r = \sqrt{0^2 + 5^2} = 5,$$

and again for the angle it's probably best to just graph it and see that $\theta = \frac{\pi}{2}$. So $z = 5e^{\frac{\pi i}{2}}$.

9. (a) The eighth roots of unity are

$$1, e^{\frac{\pi i}{4}}, e^{\frac{\pi i}{2}} = i, e^{\frac{3\pi i}{4}}, e^{\pi i} = -1, e^{\frac{5\pi i}{4}}, e^{\frac{3\pi i}{2}} = -i, e^{\frac{7\pi i}{4}}.$$

The primitive eighth roots of unity are

$$e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}.$$

(b) It must be the case that $\gcd(j, n) = 1$.

10. The base case is $n = 2$, in which case we can verify directly that

$$x^2 - 1 = (x - 1)(1 + x).$$

Now let's assume that $x^{n-1} - 1 = (x - 1)\sum_{k=0}^{n-2} x^k$. Then we have

$$\begin{aligned} x^n - 1 &= x^n - x^{n-1} + x^{n-1} - 1 \\ &= x^{n-1}(x - 1) + (x - 1)\sum_{k=0}^{n-2} x^k \\ &= (x - 1)\left(x^{n-1} + \sum_{k=0}^{n-2} x^k\right) \\ &= (x - 1)\sum_{k=0}^{n-1} x^k. \end{aligned}$$

11. To help us, let's write down the first several terms of this sequence:

$$1, 2, 4, 8, 1, 2, 4, 8, \ldots$$

We can see that the period of this sequence is 4. If we go up to 2^{32}, then the length of the sequence will be 33 (remember about the 0th term!) and the frequency will be $\frac{33}{4}$.

12. (a) i. The period is 4 and the frequency is 2

ii. After we do our rearranging as in the notes, we have

$$b_j = \frac{1}{\sqrt{8}}\left(c_0(z_j^0 + z_j) + c_1 e^{\frac{2\pi ij}{8}}(z_j^0 + z_j) + c_2 e^{\frac{4\pi ij}{8}}(z_j^0 + z_j) + c_3 e^{\frac{6\pi ij}{8}}(z_j^0 + z_j)\right),$$

with $z_j = e^{2\pi i\frac{j}{2}}$, a second root of unity. We see that if j is odd, then z_j is a primitive second root of unity ($z_j = -1$) and the sum $b_j = 0$. Otherwise, if j is even, then $z_j = 1$, so $z_j^0 + z_j = 2$, so

$$b_j = \frac{1}{\sqrt{2}}\left(c_0 + e^{\frac{\pi ij}{4}}c_1 + e^{\frac{\pi ij}{2}}c_2 + e^{\frac{3\pi ij}{4}}c_3\right),$$

which in general is "big" (at least it's not zero).
If we do two examples, we might get the following: For the sequence $1, 2, 3, 4, 1, 2, 3, 4$, we have

j	0	1	2	3	4	5	6	7
b_j	7.07	0	$-1.414 - 1.414i$	0	-1.414	0	$-1.414 + 1.414i$	0

.

For the sequence $-3, 1, -1, 5, 3, 1, -1, 5$, we have

j	0	1	2	3	4	5	6	7
b_j	5.656	0	$2.828 - 2.828i$	0	-2.828	0	$2.828 + 2.828i$	0

.

(b) i. The period is 3 and the frequency is $\frac{10}{3}$.

ii. After we do our rearranging as in the notes, we have

$$b_j = \frac{1}{\sqrt{10}}\left(c_0(z_j^0 + z_j + z_j^2 + z_j^3) + c_1 e^{\frac{2\pi i j}{10}}(z_j^0 + z_j + z_j^2) + c_2 e^{\frac{4\pi i j}{10}}(z_j^0 + z_j + z_j^2)\right),$$

where $z_j = e^{2\pi i \frac{3j}{10}}$.

We have that $\frac{3j}{10}$ is an integer when $j = 0$, as usual. After, that it is near an integer when $j = 3$ or $j = 7$ (at those points $\frac{3j}{10}$ is $\frac{9}{10}$ and $\frac{21}{10}$, respectively), or when $j = 4$ or $j = 6$ (at those points we get $\frac{12}{10}$ and $\frac{18}{10}$ respectively). So the values of b_j should be large when $j = 3, 4, 6$ and 7, which are very near "multiples" of $\frac{10}{3}$ (after all, $\frac{10}{3}$ is between 3 and 4, but closer to 3, and 3 and 6 are multiples of 3, and 4 is a multiple of 4 and 7 is a little bit larger than a multiple of 3).

The number z_j is close to being a third or a fourth root of unity (both of which would make some of the sums of roots of unity disappear) when $\frac{3j}{10}$ is close to a fraction $\frac{a}{3}$ or $\frac{b}{4}$, but a is not divisible by 3 and b is not divisible by 4. That happens basically for all of the other values of j, so all other ones should give small values for b_j.

If we do two examples, we might get the following:

For the sequence $1, 2, 3, 1, 2, 3, 1, 2, 3, 1$, we have

j	0	1	2	3	4	5	6	7	8	9
$\lvert b_j\rvert$	6.00	0.345	0.488	1.606	0.679	0.316	0.679	1.606	0.488	0.346

.

(Note that here we record absolute values since the numbers all have more digits; otherwise they don't fit neatly on one row.)

For the sequence $10, -1, 4, 10, -1, 4, 10, -1, 4, 10$, we have

j	0	1	2	3	4	5	6	7	8	9
$\lvert b_j\rvert$	15.495	1.961	2.748	8.924	3.625	1.581	3.625	8.924	2.748	1.961

.

Extra problems for graduate credit:

1. (a) The gate of parts (a) and (b) is $H \otimes H$ and the gate of part (c) is $H \otimes H \otimes H$.

(b) We prove this by induction. We already have the base case $n = 1$, by definition of the Hadamard gate H.

Now suppose that $G_n = H^{\otimes n}$ sends $|0\rangle$ to $\sum_{k=0}^{2^n-1} \frac{1}{\sqrt{2^n}}|k\rangle$, and consider the quantum gate $G_{n+1} = H \otimes G_n$.

This is, by definition of the Kronecker product, the block matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}}G_n & \frac{1}{\sqrt{2}}G_n \\ \frac{1}{\sqrt{2}}G_n & -\frac{1}{\sqrt{2}}G_n \end{pmatrix}.$$

To see the action of G_{n+1} on $|0\rangle$, we must see where this matrix sends the vector $(1\,0\,0\,\ldots 0) \in \mathbb{F}_2^{2^{n+1}}$, which is how G_{n+1} acts on $|0\rangle$. By the definition of matrix multiplication, the first 2^n entries of that vector are given by the product of $\frac{1}{\sqrt{2}}G_n$ with $(1\,0\,0\,\ldots\,0) \in \mathbb{F}_2^{2^n}$, which is how $\frac{1}{\sqrt{2}}G_n$ acts on $|0\rangle$, and the last 2^n entries of that vector are also given by the product of $\frac{1}{\sqrt{2}}G_n$ with $(1\,0\,0\,\ldots\,0) \in \mathbb{F}_2^{2^n}$. By induction, the product of G_n with $(1\,0\,0\,\ldots\,0) \in \mathbb{F}_2^{2^n}$ is a vector all of whose entries are $\frac{1}{\sqrt{2^n}}$.

Therefore, the first 2^n entries of the action of G_{n+1} on $|0\rangle$ are $\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^n}} = \frac{1}{\sqrt{2^{n+1}}}$, and the last 2^n entries are also $\frac{1}{\sqrt{2^{n+1}}}$. So indeed, G_{n+1} sends $|0\rangle$ to

$$\sum_{k=0}^{2^{n+1}-1} \frac{1}{\sqrt{2^{n+1}}}|k\rangle.$$

2. (a) Consider the sum

$$\sum_{\substack{0\le c<2^m \\ c\equiv c_0 \pmod{2^s}}} e^{\frac{2\pi i c x}{2^m}}.$$

As suggested by the hint, rather than writing $c \equiv c_0 \pmod{2^s}$, we will write $c = c_0 + 2^s j$, with $0 \le j < 2^{m-s}$. Note that if we let $0 \le j < 2^{m-s}$ in this expression, then we get exactly all $0 \le c < 2^m$ with $c \equiv c_0 \pmod{2^s}$, so that

$$\sum_{\substack{0\le c<2^m \\ c\equiv c_0 \pmod{2^s}}} e^{\frac{2\pi i c x}{2^m}} = \sum_{j=0}^{2^{m-s}-1} e^{\frac{2\pi i (c_0+2^s j) x}{2^m}}.$$

Now we have

$$\frac{2\pi i(c_0+2^s j)x}{2^m} = \frac{2\pi i c_0 x}{2^m} + \frac{2\pi i 2^s j x}{2^m} = \frac{2\pi i c_0 x}{2^m} + \frac{2\pi i 2^s x}{2^m} j,$$

so

$$e^{\frac{2\pi i(c_0+2^s j)x}{2^m}} = e^{\frac{2\pi i c_0 x}{2^m}} \cdot \left(e^{\frac{2\pi i 2^s x}{2^m}}\right)^j,$$

and our sum can be manipulated to look like

$$\sum_{\substack{0\le c<2^m \\ c\equiv c_0 \pmod{2^s}}} e^{\frac{2\pi i c x}{2^m}} = e^{\frac{2\pi i c_0 x}{2^m}} \sum_{j=0}^{2^{m-s}-1} \left(e^{\frac{2\pi i 2^s x}{2^m}}\right)^j.$$

This is indeed the geometric sum $\sum_{j=0}^{n} r^j$, with $n = 2^{m-s}-1$ and $r = e^{\frac{2\pi i 2^s x}{2^m}}$. Now if $r = 1$, then it is straightforward to see that the sum is $n+1$ (since are just adding 1 $n+1$ times). If $r = -1$, then the sum depends on whether n is even or

odd (it is either 0 if n is odd, since we end on a -1 term, or it is 1 if n is even, since we end on a 1 term). Otherwise, if $|r| \neq 1$, the sum is $\frac{1-r^{n+1}}{1-r}$.

We have that $r = e^{\frac{2\pi i 2^s x}{2^m}} = 1$ exactly when $\frac{2^s x}{2^m}$ is an integer, which is to say that $2^s x$ is divisible by 2^m, or x is divisible by 2^{m-s}, or $x \equiv 0 \pmod{2^{m-s}}$. Then the sum is

$$\sum_{\substack{0 \leq c < 2^m \\ c \equiv c_0 \pmod{2^s}}} e^{\frac{2\pi i c x}{2^m}} = e^{\frac{2\pi i c_0 x}{2^m}} 2^{m-s},$$

as claimed.

Suppose now that x is such that $r = e^{\frac{2\pi i 2^s x}{2^m}} = -1$. In that case, since $2^{m-s} - 1$ is certainly odd, the sum is 0, again as claimed.

Finally, in all other cases we have

$$\sum_{\substack{0 \leq c < 2^m \\ c \equiv c_0 \pmod{2^s}}} e^{\frac{2\pi i c x}{2^m}} = e^{\frac{2\pi i c_0 x}{2^m}} \frac{1 - \left(e^{\frac{2\pi i 2^s x}{2^m}}\right)^{2^{m-s}}}{1 - e^{\frac{2\pi i 2^s x}{2^m}}},$$

but

$$\left(e^{\frac{2\pi i 2^s x}{2^m}}\right)^{2^{m-s}} = e^{2\pi i x} = \left(e^{2\pi i}\right)^x = 1^x = 1,$$

since x is an integer. Therefore

$$\sum_{\substack{0 \leq c < 2^m \\ c \equiv c_0 \pmod{2^s}}} e^{\frac{2\pi i c x}{2^m}} = e^{\frac{2\pi i c_0 x}{2^m}} \frac{1 - \left(e^{\frac{2\pi i 2^s x}{2^m}}\right)^{2^{m-s}}}{1 - e^{\frac{2\pi i 2^s x}{2^m}}} = e^{\frac{2\pi i c_0 x}{2^m}} \frac{1 - 1}{1 - e^{\frac{2\pi i 2^s x}{2^m}}} = 0,$$

again as claimed.

3. I did only two examples of each but I did more in the notes!