Math 259 - Spring 2019
Homework 2

This homework is due on Monday, February 11. Note that while you may use a calculator to do these problems, you should be able to do them by hand for the quiz.

1. Write each of the following base-10 numbers in binary:

   (a) 2

   (b) 6

2. Write each of the following binary numbers in base-10:

   (a) 1000

   (b) 101

3. **In hindsight this problem didn't work, so please omit it.**

4. For each of the following complex numbers z, compute $|z|$, its absolute value:

   (a) $z = 5$

   (b) $z = -4$

   (c) $z = 2i$

   (d) $z = 3 + 4i$

   (e) $z = -1 + 3i$

   (f) $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

5. Determine if each of the following is an n-qbit superposition. If not, state why not. If so, say what n is, and compute the probability of observing each state/answer.

   (a) $\dfrac{3i}{5}|0\rangle + \dfrac{4}{5}|1\rangle$

   (b) $\dfrac{1}{2}|0\rangle + \dfrac{1}{2}|1\rangle$

   (c) $\displaystyle\sum_{k=0}^{7} \frac{1}{2\sqrt{2}}|k\rangle$

   (d) $\displaystyle\sum_{k=0}^{2^n-1} \frac{1}{\sqrt{2^n}}|k\rangle$

6. In class on Friday you will see the (1-qbit) Hadamard gate. In this problem we will consider generalizations of this gate. For reasons that will become apparent, we wish to write a quantum gate that sends the 2-qbit superposition $|00\rangle$ to the superposition

$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

   (a) Show that we can obtain this by applying the Hadamard gate to each qbit separately. In other words, what happens if you take the 1-qbit superposition $|0\rangle$ and call it the "first" qbit, put it through the Hadamard gate, then take another 1-qbit superposition $|0\rangle$ and call it the "second" qbit, and put it through the Hadamard gate? What is the probability that you observe $|00\rangle$ as the answer (i.e. 0 on the reading from the "first" qbit and 0 on the reading from the "second" qbit)? The probability that you observe $|01\rangle$ as the answer (i.e. 0 on the reading from the "first" qbit and 1 on the reading from the "second" qbit)? The probability that you observe $|10\rangle$? The probability that you observe $|11\rangle$?

(b) Write the full permutation matrix for this quantum gate. (In other words, compute what it does to the other states $|01\rangle$, $|10\rangle$ and $|11\rangle$ and assemble in a matrix appropriately.)

(c) Explain how you could obtain a quantum gate that sends $|000\rangle$ to

$$\sum_{k=0}^{7} \frac{1}{2\sqrt{2}}|k\rangle.$$

7. The SWAP gate is a gate that takes a 2-bit number and outputs the 2-bit number that has its bits swapped (so if you input 01, it would output 10 for example).

   (a) Write a full table of inputs/outputs (or truth table) for this gate, like we did in class for other gates.

   (b) Write a permutation matrix for this gate.

   (c) Is this gate a quantum gate?

8. For each of the following complex numbers, if they are of the form $z = x + iy$, please convert to the form $z = re^{i\theta}$. If they are of the form $z = re^{i\theta}$, please convert to the form $z = x + iy$.

   (a) $z = 4e^{\frac{2\pi i}{3}}$

   (b) $z = -1 - i$

   (c) $z = 2e^{\frac{7\pi i}{4}}$

   (d) $z = 5i$

9. (a) Please list all 8th roots of unity. Which ones are primitive?

   (b) Let $\zeta = e^{\frac{2\pi ij}{n}}$ be an nth root of unity. What is the condition that j must satisfy so that ζ is a primitive nth root of unity?

10. Use induction to show that for all $n \geq 2$,

$$x^n - 1 = (x - 1)\sum_{k=0}^{n-1} x^k.$$

11. (adapted from TW Section 19.4 problem 1) Consider the sequence $2^0, 2^1, 2^2, \ldots \pmod{15}$. What is the period of this sequence? If we go all the way to 2^{32} and stop there, what is the frequency of this sequence?

12. For each of the following sequences, do the following:

i. Compute the period and the frequency; and

ii. Either (or ideally both, and graduate students must do both)

- use an argument similar to the one we used in class to predict where the discrete Fourier transform will have large and small values; or
- choose at least four concrete example values for $c_0, c_1, c_2, \ldots$, use a computer to compute the discrete Fourier transform, and note where you obtain large and small values (in absolute value).

In both cases, relate your findings to the period/frequency of the original sequence.

(a) $c_0, c_1, c_2, c_3, c_0, c_1, c_2, c_3$

(b) $c_0, c_1, c_2, c_0, c_1, c_2, c_0, c_1, c_2, c_0$

Extra problems for graduate credit:

1. Let H be the Hadamard gate. For this problem, please first read up on the Kronecker product of two matrices

   (a) Write the gate of undergraduate problem 6 (a) and (b) as a Kronecker product of Hadamard gates. What about the gate of undergraduate problem 6 (c)?

   (b) Show that $G_n = H^{\otimes n}$ sends the n-qbit superposition $|0\rangle$ to

$$\sum_{k=0}^{2^n-1} \frac{1}{\sqrt{2^n}} |k\rangle.$$

2. (TW Section 19.4 problem 2)

   (a) Let $0 < s \leq m$. Fix and integer c_0 with $0 \leq c_0 < 2^s$. Show that

$$\sum_{\substack{0 \leq c < 2^m \\ c \equiv c_0 \pmod{2^s}}} e^{\frac{2\pi i c x}{2^m}} = 0$$

   if $x \not\equiv 0 \pmod{2^{m-s}}$ and that

$$\sum_{\substack{0 \leq c < 2^m \\ c \equiv c_0 \pmod{2^s}}} e^{\frac{2\pi i c x}{2^m}} = 2^{m-s} e^{\frac{2\pi i x c_0}{2^m}}$$

   if $x \equiv 0 \pmod{2^{m-s}}$.
   Hint: Write $c = c_0 + 2^s j$ with $0 \leq j < 2^{m-s}$; after factoring you should recognize a geometric sum.

   (b) Suppose that $a_0, a_1, \ldots, a_{2^m-1}$ is a sequence of length 2^m such that $a_k = a_{k+2^s j}$ for all j, k. Show that the Fourier transform b_ℓ of this sequence is 0 whenever $\ell \not\equiv 0 \pmod{2^{m-s}}$. Conclude that if the period of a sequence is a divisor of 2^m, then all the nonzero values of the DFT occur at multiples of the frequency. (What is the frequency here?)

3. Make sure that you did both options in Undergraduate Problem 12.