Name:

Problem 1: *Please give all solutions to the quadratic congruence*

$$x^2 \equiv 1 \pmod{72}.$$

Solution: We have that $72 = 8 \cdot 9$, so we must solve

$$x^2 \equiv 1 \pmod{8} \quad \text{and} x^2 \equiv 1 \pmod{9}.$$

These are both straightforward: $x^2 \equiv 1 \pmod 8$ has solutions $x \equiv 1, 3, 5, 7 \pmod 8$, and $x^2 \equiv 1 \pmod 9$ has solutions $x \equiv 1, 8 \pmod 9$ (since 9 is a power of an odd prime, we know there are only two solutions).
Therefore all we must do is to form all eight pairs of solutions modulo 8 and modulo 9 to compute the eight solutions to $x^2 \equiv 1 \pmod{72}$. We first compute what the form of the solution will be. Here we have that $m_1 = 8$ and $m_2 = 9$, so that $M_1 = 9$ and $x_1 \equiv 9^{-1} \equiv 1^{-1} \equiv 1 \pmod 8$, and $M_2 = 8$ and $x_2 \equiv 8^{-1} \equiv (-1)^{-1} \equiv -1 \pmod 9$.
Therefore, the solution that correspond to the pair

$$x \equiv a_1 \pmod 8, \quad x \equiv a_2 \pmod 9$$

is

$$x \equiv 9a_1 - 8a_2 \pmod{72}.$$

It follows that the eight solutions are

$$\begin{gathered}
x \equiv 9 \cdot 1 - 8 \cdot 1 \equiv 1 \pmod{72}, \\
x \equiv 9 \cdot 1 - 8 \cdot 8 \equiv -55 \equiv 17 \pmod{72}, \\
x \equiv 9 \cdot 3 - 8 \cdot 1 \equiv 19 \pmod{72}, \\
x \equiv 9 \cdot 3 - 8 \cdot 8 \equiv -37 \equiv 35 \pmod{72}, \\
x \equiv 9 \cdot 5 - 8 \cdot 1 \equiv 37 \pmod{72}, \\
x \equiv 9 \cdot 5 - 8 \cdot 8 \equiv -19 \equiv 53 \pmod{72}, \\
x \equiv 9 \cdot 7 - 8 \cdot 1 \equiv 55 \pmod{72}, \\
x \equiv 9 \cdot 7 - 8 \cdot 8 \equiv -1 \equiv 71 \pmod{72}.
\end{gathered}$$