Name:

Problem 1: *Please give all solutions to the quadratic congruence*

$$x^2 \equiv 21 \pmod{25}.$$

Solution: We have that $(21, 25) = 1$ and $25 = 5^2$, so we can use our lifting technique. We first solve $x^2 \equiv 21 \equiv 1 \pmod{5}$. This has solution $x_0 \equiv -1 \equiv 4 \pmod{5}$.
We now lift this solution to $\mathbb{Z}/25\mathbb{Z}$. The lifting equation is

$$x_1 = 4 + 5y_0$$

and we wish to solve the equation

$$x_1^2 \equiv 21 \pmod{25}.$$

Plugging the first equation into the second we get:

$$\begin{aligned}
(4 + 5y_0)^2 &\equiv 21 \pmod{25} \\
16 + 40y_0 + 25y_0^2 &\equiv 21 \pmod{25} \\
16 + 15y_0 &\equiv 21 \pmod{25} \\
15y_0 &\equiv 5 \pmod{25} \\
3y_0 &\equiv 1 \pmod{5} \\
y_0 &\equiv 2 \pmod{5}.
\end{aligned}$$

Therefore we get the solution $x_1 \equiv 4 + 5 \cdot 2 \equiv 14 \pmod{25}$, and the other solution is $-x_1 \equiv -14 \equiv 11 \pmod{25}$.