Name:

Problem 1: *What are the last two digits of the integer*

$$48^{48}?$$

Solution: The last two digits of a positive integer are exactly the least residue of this integer modulo 100. Therefore we wish to compute $48^{48} \pmod{100}$. Because $(48, 100) \neq 1$, we cannot use Euler's Theorem directly.
Instead, we will compute $48^{48} \pmod 4$ and $48^{48} \pmod{25}$ separately, and use the Chinese Remainder Theorem to get the answer.
We have that $48 \equiv 0 \pmod 4$, and therefore $48^{48} \equiv 0^{48} \equiv 0 \pmod 4$.
Now $(25, 48) = 1$, so we can apply Euler's Theorem here. To make our life easier, we note that $48 \equiv -2 \pmod{25}$, so $48^{48} \equiv (-2)^{48} \pmod{25}$. Furthermore, we have that $\phi(25) = 25 - 5 = 20$, and $48 = 2 \times 20 + 8$, so

$$\begin{aligned}
(-2)^{48} &\equiv (-2)^{20\times 2+8} \pmod{25} \\
&\equiv ((-2)^{20})^2 \cdot (-2)^8 \pmod{25} \\
&\equiv 1^2 \cdot (-2)^8 \pmod{25} \\
&\equiv (-2)^8 \pmod{25} \\
&\equiv ((-2)^4)^2 \pmod{25} \\
&\equiv 16^2 \pmod{25} \\
&\equiv (-9)^2 \pmod{25} \\
&\equiv 81 \pmod{25} \\
&\equiv 6 \pmod{25}
\end{aligned}$$

Therefore we are looking for a number that is both $0 \pmod 4$ and $6 \pmod{25}$. Rather than use the Chinese Remainder Theorem algorithm, we can use a simple observation that follows from it: There are only four lifts of $6 \pmod{25}$ to $\mathbb{Z}/100\mathbb{Z}$: $6, 31, 56$ and $81 \pmod{100}$. Only one of these is $0 \pmod 4$, and that is $56 \pmod{100}$. Therefore

$$48^{48} \equiv 56 \pmod{100},$$

and its last two digits are 56.