Math 255: Spring 2018
Practice Final Exam

NAME:

Time: **2 hours and 45 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ____

Problem	Value	Score
1	4	
2	5	
3	8	
4	12	
5	8	
6	15	
7	8	
8	8	
9	8	
10	8	
11	8	
12	8	
GC	8	
TOTAL	100 (or 108)	

Problem 1 : (4 points) Compute $8^{-1} \pmod{29}$.

Problem 2 : (5 points) What is the (multiplicative) order of 2 modulo 11?

Problem 3 : (8 points) Give all **positive** integer solutions to the equation $39x + 51y = 3$.

Problem 4 : (12 points) Solve the following equations. For each equation, give **all** distinct solutions (if there are more than one) and be sure to clearly indicate which ring the solutions belong to.

a) $4x \equiv 5 \pmod 6$

b) $13x \equiv 1 \pmod{17}$

c) $15x \equiv 10 \pmod{25}$

Problem 5 : (8 points) Solve the following systems of equations. Be sure to give **all** distinct solutions (if there are more than one) and to clearly indicate which ring the solution(s) belong to.

a) $x \equiv 1 \pmod 2, \quad 3x \equiv 1 \pmod 5, \quad 4x \equiv 5 \pmod 7$

b) $2x \equiv 4 \pmod{24}, \quad 3x \equiv 8 \pmod{10}, \quad 2x \equiv 22 \pmod{45}$

Problem 6 : (15 points) Give all solutions to the following quadratic congruences.

a) $x^2 \equiv 65 \pmod{128}$

b) $x^2 \equiv 23 \pmod{121}$

c) $x^2 \equiv 9 \pmod{20}$

Problem 7 : (8 points) Let a be an integer. Show that a and a^{4n+1} always have the same last digit.

Problem 8 : (8 points) In 1644, Mersenne asked for a positive integer with 60 distinct divisors. Find one such integer that is smaller than 10,000.

Problem 9 : (8 points) Let n be a positive integer with $6|n$. Show that $\phi(n) \leq \frac{n}{3}$.

Problem 10 : (8 points) Let $a > 1$ and $m > 1$ be integers, and let p be a prime. Show that if $p \equiv a \pmod{m}$, then $(a, m) = 1$ or $a = p$.

Problem 11 : (8 points) Let p be a prime. Prove that

$$(a+b)^p \equiv a^p + b^p \pmod{p}.$$

Note: You can use this fact on the Final Exam. It is sometimes called the Freshman's Dream, for reasons you can perhaps imagine.

Problem 12 : (8 points) Let $p \geq 5$ be a prime, and let a have order 4 modulo p. What is the least residue of $(a+1)^4 \pmod{p}$?

Extra problem for graduate credit:

Problem 13 : (8 points) Throughout, let p be an odd prime and a be an integer with $(a, p) = 1$. It might help to remember that in this case p has a primitive root.

a) Show that if $x^2 \equiv a \pmod{p}$ has a solution, then $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$.

b) Show that the converse is also true: If $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$, then $x^2 \equiv a \pmod{p}$ has a solution.