Math 255 - Spring 2018
Homework 5 Solutions

1. It might be obvious that $(23, 14) = 1$, and therefore by Theorem 1 of Section 3, this equation has integer solutions (since 1 divides 2).

   We begin by using the Euclidean Algorithm to find integer solutions x_0, y_0 to the equation $23x_0 + 14y_0 = 1$: We begin with

$$\begin{aligned} 23 &= 14 + 9 \\ 14 &= 9 + 5 \\ 9 &= 5 + 4 \\ 5 &= 4 + 1 \\ 4 &= 4 \cdot 1. \end{aligned}$$

   Therefore we have

$$\begin{aligned} 1 &= 5 - 4 \\ 4 &= 9 - 5 \\ 5 &= 14 - 9 \\ 9 &= 23 - 14. \end{aligned}$$

   Back-substituting, we get

$$\begin{aligned} 1 &= 5 - 4 \\ &= 5 - (9 - 5) \\ &= 5 - 9 + 5 \\ &= 2 \cdot 5 - 9 \\ &= 2 \cdot (14 - 9) - 9 \\ &= 2 \cdot 14 - 2 \cdot 9 - 9 \\ &= 2 \cdot 14 - 3 \cdot 9 \\ &= 2 \cdot 14 - 3 \cdot (23 - 14) \\ &= 2 \cdot 14 - 3 \cdot 23 + 3 \cdot 14 \\ &= 5 \cdot 14 - 3 \cdot 23. \end{aligned}$$

   From this we obtain the solution $x_0 = -3$ and $y_0 = 5$ to the equation $23x_0 + 14y_0 = 1$ (which we note is NOT the equation we are trying to solve!)

   Since $2 = 2 \cdot 1$, we can multiply both sides of $23 \cdot (-3) + 14 \cdot 5 = 1$ by 2 to obtain

$$2(23 \cdot (-3) + 14 \cdot 5) = 2,$$

or, using distributivity and commutativity,

$$23 \cdot (-6) + 14 \cdot 10 = 2,$$

which gives us the particular solution $x_p = -6$ and $y_p = 10$ to the equation $23x+14y = 2$ (and this IS the equation we are trying to solve!)

Now it remains to apply Theorem 1 of Section 3, with $a = 23$, $b = 14$ and $(a, b) = 1$ to write that all integer solutions are given by

$$\begin{aligned} x &= -6 + 14t \\ y &= 10 - 23t, \end{aligned}$$

with t ranging over all integers.

2. In this problem, let x be the number of calves bought by the farmer, y be the number of lambs bought by the farmer, and z be the number of piglets bought by the farmer. Note that x, y and z are all positive integers since the farmer buys at least one animal of each species.

   Translating the problem into math gives us the equations

   $$x + y + z = 100$$

   and

   $$120x + 50y + 25z = 4000.$$

   Substituting $z = 100 - x - y$ into the second equation, we obtain

   $$95x + 25y = 1500,$$

   which is an equation of the type we have been studying, and for which we can therefore find all positive integer solutions.

   We first use the Euclidean algorithm to determine $(95, 25)$ and ascertain that it divides 1500: We have

   $$\begin{aligned} 95 &= 25 \cdot 3 + 20 \\ 25 &= 20 + 5 \\ 20 &= 5 \cdot 4. \end{aligned}$$

   Therefore $(95, 25) = 5$, and since 5 does divide 1500, there will be integer solutions to this equation.

   We now back-substitute to solve the equation $95x_0 + 25y_0 = 5$:

   $$\begin{aligned} 5 &= 25 - 20 \\ &= 25 - (95 - 3 \cdot 25) \\ &= 25 - 95 + 3 \cdot 25 \\ &= 4 \cdot 25 - 95. \end{aligned}$$

This gives us the solution $x_0 = -1$ and $y_0 = 4$, but we stress that this is for the equation $95x_0 + 25y_0 = 5$, which is not the equation we are trying to solve.

We now multiply each side by 300 to obtain a particular solution to the equation we do care about: First we get

$$300(4 \cdot 25 - 95) = 1500,$$

and using distributivity and commutativity we get

$$25 \cdot 1200 + 95 \cdot (-300) = 1500.$$

Therefore we have the particular solutions $x_p = -300$ and $y_p = 1200$ to the equation $95x + 25y = 1500$.

To get all solutions that are positive integers, we first write down all solutions using Theorem 1 of Section 3, with $a = 95$, $b = 25$ and $(a, b) = 5$. This gives us the solutions

$$\begin{aligned} x &= -300 + 5t \\ y &= 1200 - 19t, \end{aligned}$$

with $t \in \mathbb{Z}$.

To find the positive solutions we solve

$$-300 + 5t > 0 \quad \text{and} \quad 1200 - 19t > 0.$$

The first equation gives us $t > 60$, and the second equation gives us $t < \frac{1200}{19} \approx 63.15$. Therefore the possibilities for t, if t is to be an integer, are $t = 61, 62$, and 63.

To finally answer the problem, we recall that there was a third variable z, and that this variable should also be positive. Therefore we obtain the pairs (x, y) associated to each t, and compute the associated z using $z = 100 - x - y$.

If $t = 61$, then $x = -300+305 = 5$ and $y = 1200-1159 = 41$, and $z = 100-5-41 = 54$. This is an acceptable answer.

If $t = 62$, then $x = -300+310 = 10$ and $y = 1200-1178 = 22$, and $z = 100-10-22 = 68$. This is also an acceptable answer.

It $t = 63$, then $x = -300+315 = 15$ and $y = 1200-1197 = 3$, and $z = 100-15-3 = 82$. This is also an acceptable answer.

(Note that it is a coincidence that these all give acceptable values of z; there very well could have been a pair (x, y) giving a negative value of z if the numbers had turned out differently.)

Therefore the farmer either bought 5 calves, 41 lambs and 54 piglets; or 10 calves, 22 lambs and 68 piglets; or 15 calves, 3 lambs and 82 piglets.

3. Since n is a composite number, there exist integers d_1 and d_2 with $1 < d_1 \leq d_2 < n$ and $n = d_1 d_2$ (in other words, n has divisors other than 1 and itself). There are two possibilities: Either $d_1 \neq d_2$, or $d_1 = d_2$. We tackle each possibility in turn and show that n divides $(n-1)!$ in either case.

   If $d_1 \neq d_2$, then the product $(n-1)!$ contains both the integer d_1 and the integer d_2 as separate factors (since they are distinct, positive, and both less than or equal to $n-1$). Therefore since the rest of the product is an integer, $n = d_1 d_2$ divides $(n-1)!$.

   If $d_1 = d_2$, this argument does not work (and there are composite integers $n > 4$ for which this is the only option: $n = p^2$ for p a prime). For simplicity let's write $n = d^2$, with $1 < d < n$.

   Since $n > 4$, then $d > 2$. To find the other factor of d inside $(n-1)!$, consider the integer $2d$. Since $d > 2$, it follows that $d^2 > 2d$, and since $n = d^2$, then $n > 2d$. Therefore d and $2d$ are two distinct integers contained strictly between 1 and n, and therefore they both appear separately in the product $(n-1)!$. By an argument similar to the one above, we conclude that $d \cdot 2d = 2d^2$ divides $(n-1)!$. Since $n = d^2$ divides $2d^2$, by Exercise 2 in Section 1, we may conclude that n divides $(n-1)!$.

   Note that if $n = 4$ (which is the only composite number not covered by this problem), then the conclusion is false: $(4-1)! = 3! = 6$ and 4 does not divide 6. Therefore it is absolutely necessary for the argument to work that $d > 2$; otherwise there is not enough "space" for both d and $2d$ to be strictly between 1 and n.

4. Let us first establish the following: If both p and q are greater than $n^{1/4}$ then $pq > n^{1/2}$, and therefore $\frac{n}{pq} < \frac{n}{n^{1/2}} = n^{1/2}$. Therefore $\frac{n}{pq}$ is a small factor of n.

   Therefore the question roughly asks if an integer n having large prime factors affects whether its small factors are prime. The assertion as stated is false: The large prime factors of an integer n do not affect the small factors.

   We now attempt to build a counter-example. Let $p = 7$ and $q = 11$, so these are relatively large. Then let $\frac{n}{pq} = 6$, so that it is small but not prime. Then $n = 6 \cdot 7 \cdot 11 = 462$, and $n^{1/4} \approx 4.63$. Then both p and q are greater than $n^{1/4}$, as required, and $\frac{n}{pq} = 6$ is not prime, therefore disproving the assertion.