Math 255 - Spring 2018
Homework 4 Solutions

1. We begin by collecting our definitions. Since $c|ab$, there is an integer s such that $ab = cs$. Furthermore, since d is the greatest common divisor of c and a, we have:

   - d divides c (so there is an integer t with $c = dt$) and d divides a (so there is an integer r with $a = dr$), and
   - whenever $e|c$ and $e|a$, then $e \leq d$.

   (We now pause for a second to consider our goal: Our goal is to show that c divides db, or in other words that there is an integer u such that $db = cu$. The plan is to go from the equation $ab = cs$ to this equation, somehow. Let's see what must be true for that to happen.

   We begin by plugging $a = dr$ into the equation $ab = cs$:

   $$(dr)b = cs.$$

   From this, it seems that if we could divide both sides by r and get $\frac{s}{r}$ to be an integer, that would be our integer u and we would be done. Therefore our goal is to show that r divides s, or in other words that there is an integer u such that $s = ru$.)

   We begin by establishing that $(r,t) = 1$: Indeed, since $r = \frac{a}{d}$ and $t = \frac{c}{d}$, where $d = (a,c)$, by Section 1, Theorem 1, we have that $(r,t) = \left(\frac{a}{d}, \frac{c}{d}\right) = 1$.

   We now plug in $c = dt$ into the equation $(dr)b = cs$ and get

   $$(dr)b = (dt)s.$$

   Dividing both sides by d, we obtain

   $$rb = ts.$$

   Therefore, r divides ts, since b is an integer. But $(r,t) = 1$, so we conclude that $r|s$.

   Therefore we do have that there is an integer u with $s = ru$, and therefore plugging this into the equation $(dr)b = cs$, we get

   $$(dr)b = c(ru),$$

   and dividing both sides by r yields $db = cu$ with u an integer, from which it follows that c divides db.

2. According to the Bonus Proposition from class, for each a it suffices to show that there exist integers x and y such that

   $$(2a+1)x + (9a+4)y = 1.$$

One way to see if we can make this happen is to treat a as an indeterminate. If we can solve the equation for x and y when a is an indeterminate, then by substituting each value of a we will obtain values of x and y that satisfy the equation.

In that case, if a is an indeterminate, we may assume that a and 1 are linearly independent and therefore we get the system of equations

$$\begin{aligned} 2x + 9y &= 0 \\ x + 4y &= 1, \end{aligned}$$

where the first equation comes from equating the coefficient of a on each side and the second equation comes from equating the constant term on each side. Substituting $x = 1 - 4y$ into the first equation we get

$$2 + y = 0,$$

so $y = -2$ and $x = 9$.

Therefore, no matter what a is, there is always the integer solution $x = 9$ and $y = -2$ to the equation $(2a+1)x + (9a+4)y = 1$ (try it at home with some values of a!). Therefore $(2a+1, 9a+4) = 1$.

3. Let's do it.

   We first do the Euclidean Algorithm:

$$\begin{aligned} 299 &= 247 + 52 \\ 247 &= 52 \cdot 4 + 39 \\ 52 &= 39 + 13 \\ 39 &= 13 \cdot 3. \end{aligned}$$

   (We note that since $(299, 247) = 13$, there is indeed a solution!)

   Next we solve for each remainder:

$$\begin{aligned} 13 &= 52 - 39 \\ 39 &= 247 - 52 \cdot 4 \\ 52 &= 299 - 247. \end{aligned}$$

   Finally we back-solve. We first plug the second equation into the first and collect like terms:

$$\begin{aligned} 13 &= 52 - (247 - 52 \cdot 4) \\ &= 52 - 247 + 4 \cdot 52 \\ &= 5 \cdot 52 - 247. \end{aligned}$$

Now we plug the last equation into this equation:

$$\begin{aligned} 13 &= 5 \cdot 52 - 247 \\ &= 5 \cdot (299 - 247) - 247 \\ &= 5 \cdot 299 - 5 \cdot 247 - 247 \\ &= 5 \cdot 299 - 6 \cdot 247. \end{aligned}$$

This is what we wanted, this gives us the solution $x = 5$ and $y = -6$.

4. We note that since $c \mid (a+b)$, there is $r \in \mathbb{Z}$ such that $a + b = cr$.

   Suppose that $(a, c) = d$. Then there are s and t integers such that $a = ds$ and $c = dt$. Substituting this into the equation $a + b = cr$, we get

   $$ds + b = dtr.$$

   Solving for b, we get
   $$b = dtr - ds = d(tr - s).$$

   Since $tr - s$ is an integer, d divides b. Therefore d is a common divisor of a and b. However, by assumption $(a, b) = 1$, therefore $d \leq 1$. Since $d \geq 1$ since it is a greatest common divisor, we may conclude that $d = 1$.

   A similar argument, replacing the roles of a and b, shows that $(b, c) = 1$ as well.